

Research Article **Double Controlled Quasi-Metric Type Spaces and Some Results**

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Abdeljawad et al. (2018) introduced a new concept, named double controlled metric type spaces, as a generalization of the notion of extended *b*-metric spaces. In this paper, we extend their concept and introduce the concept of double controlled quasi-metric type spaces with two incomparable functions and prove some unique fixed point results involving new types of contraction conditions. Also, we introduce the concept of $\alpha - \mu - k$ double controlled contraction and prove some related fixed point results. We give several examples to show that our results are the proper generalization of the existing works.

1. Introduction and Preliminaries

The theory of fixed points takes an important place in the transition from classical analysis to modern analysis. One of the most remarkable work on fixed point theory was done by Banach [1]. Various generalizations of Banach fixed point theorem were made by numerous mathematicians, see [2-4]. One of the abstraction of the metric spaces is the quasi-metric space that was introduced by Wilson [5]. The commutativity condition does not hold in general in quasi-metric spaces. Several authors used this concept to prove some fixed point results, see [6-18]. On the other hand, Bakhtin [19] and Czerwik [20] established the idea of *b*-metric spaces. Lateral, many authors got several fixed point results, for instance, see [21-25]. Kamran et al. [26] introduced a new idea to generalized b-metric spaces, named as extended b-metric spaces, see also [27–30]. They replaced the parameter $b \ge 1$ in the triangle inequality by the control function $\theta: G \times G \longrightarrow [1, \infty)$. Nurwahyu [31] introduced dislocated quasi-extended b-metric space and obtained several fixed point results. Mlaiki et al. [32] generalized the triangle inequality in *b*-metric spaces by using control function in a different style and introduced controlled

metric type spaces. Recently, Abdeljawad et al. [33] generalized the idea of extended *b*-metric spaces as well as controlled metric type spaces and introduced double controlled metric type spaces. They replaced the control function θ in triangle inequality by two control functions α and μ . Now, we recall some basic definitions and examples that will be used in this paper.

Definition 1 (see [33]). Given noncomparable functions $\alpha, \mu: G \times G \longrightarrow [1, +\infty)$. If $f: G \times G \longrightarrow [0, +\infty)$ satisfies

- (q1) $f(v, \gamma) = 0$ if and only if $v = \gamma$
- (q2) $f(v, \gamma) = f(\gamma, v)$
- (q3) $f(v, \gamma) \le \alpha(v, e) f(v, e) + \mu(e, \gamma) f(e, \gamma)$, for all $v, \gamma, e \in G$

Then, *f* is called a double controlled metric type with the functions α , μ and the pair (*G*, *f*) is called double controlled metric type space with the functions α , μ .

Theorem 1 (see [33]). Let (G, f) be a complete double controlled metric type space with the functions $\alpha, \mu: G \times G \longrightarrow [1, +\infty)$ and let $T: G \longrightarrow G$ be a given mapping. Suppose that the following conditions are satisfied.

There exists $k \in (0, 1)$ such that

$$fT(\dot{c}), T(y) \le kf(\dot{c}, y), \quad \text{for all } \dot{c}, y \in G.$$
(1)

For $v_0 \in G$, choose $v_p = T^p v_0$. Assume that

$$\sup_{m \ge 1} \lim_{i \longrightarrow +\infty} \frac{\alpha(v_{i+1}, v_{i+2})}{\alpha(v_i, v_{i+1})} \mu(v_{i+1}, v_m) < \frac{1}{k}.$$
 (2)

In addition, for every $v \in G$, we have

 $\lim_{p \longrightarrow +\infty} \alpha(v, v_p) \text{ and } \lim_{p \longrightarrow +\infty} \mu(v_p, v) \text{ exist and are finite.}$ (3)

Then, T has a unique fixed point $v^* \in G$.

Definition 2. Given noncomparable functions $\alpha, \mu: G \times G \longrightarrow [1, +\infty)$. If $f: G \times G \longrightarrow [0, +\infty)$ satisfies

(Q1)
$$f(v, \gamma) = 0$$
 if and only if $v = \gamma$
(Q2) $f(v, \gamma) \le \alpha(v, e) f(v, e) + \mu(e, \gamma) f(e, \gamma)$, for all $v, \gamma, e \in G$

Then, f is called a double controlled quasi-metric type with the functions α and μ and (G, f) is called a double controlled quasi-metric type space. If $\mu(e, \gamma) = \alpha(e, \gamma)$, then (G, f) is called a controlled quasi-metric type space.

Remark 1. Any quasi-metric space, double controlled metric type space, and controlled quasi-metric type space are also double controlled quasi-metric type space, but the converse is always not true (see Examples 1–3).

Example 1. Let $G = \{0, 1, 2\}$. Define $f: G \times G \longrightarrow [0, +\infty)$ by f(0, 1) = 4, f(0, 2) = 1, f(1, 0) = 3 = f(1, 2), f(2, 0) = 0, f(2, 1) = 2, and f(0, 0) = f(1, 1) = f(2, 2) = 0. Define $\alpha, \mu: G \times G \longrightarrow [1, +\infty)$ as $\alpha(0, 1) = \alpha(1, 0) = \alpha(1, 2) = 1$, $\alpha(0, 2) = 5/4$, $\alpha(2, 0) = 10/9$, $\alpha(2, 1) = 20/19$, $\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = 1$, $\mu(0, 1) = \mu(1, 0) = \mu(0, 2) = \mu(1, 2) = 1$, $\mu(2, 0) = 3/2$, $\mu(2, 1) = 11/8$, and $\mu(0, 0) = \mu(1, 1) = \mu(2, 2) = 1$.

To show that the usual triangle inequality in quasi-metric is not satisfied. Let v = 0, e = 2, and $\gamma = 1$, then we have

$$f(0,1) = 4 > 3 = f(0,2) + f(2,1), \tag{4}$$

this shows that f is a double controlled quasi-metric type for all $v, y, e \in G$, but it is not a controlled quasi-metric type. Indeed,

$$f(0,1) = 4 > \frac{255}{76} = \alpha(0,2)f(0,2) + \alpha(2,1)f(2,1).$$
 (5)

Also, it is not a double controlled metric type space because we have

$$f(0,1) = 4 = \alpha(0,2)f(0,2) + \mu(2,1)f(2,1) \neq f(1,0).$$
(6)

Definition 3. Let (G, f) be a double controlled quasi-metric type space with two functions. A sequence $\{v_p\}$ is convergent to some v in G if and only if $\lim_{p \to +\infty} f(v_p, v) = \lim_{p \to +\infty} f(v, v_p) = 0$.

Definition 4. Let (G, f) be a double controlled quasi-metric type space with two functions:

- (i) The sequence $\{v_p\}$ is a left Cauchy if and only if for every $\varepsilon > 0$, we obtain a positive integer p_{ε} such that $f(v_m, v_p) < \varepsilon$, for all $p > m > p_{\varepsilon}$ or $\lim_{p,m \to +\infty} f(v_m, v_p) = 0$
- (ii) The sequence {v_p} is a right Cauchy if and only if for every ε > 0, we obtain a positive integer p_ε such that f (v_m, v_p) < ε, for all m > p > p_ε or lim_{p,m→+∞}f (v_m, v_p) = 0
- (iii) The sequence $\{v_p\}$ is a dual Cauchy if and only if it is left Cauchy as well as right Cauchy

Definition 5. Let (G, f) be a double controlled quasi-metric type space. Then, (G, f)(G, f) is

- (i) Left complete if and only if each left Cauchy sequence in *G* is convergent
- (ii) Right complete if and only if each right Cauchy sequence in *G* is convergent
- (iii) Dual complete if and only if every left Cauchy as well as right Cauchy sequence in G is convergent

Note that each dual complete double controlled quasimetric type space is left complete as well as right complete, but converse is not true in general.

2. Main Results

In this section, we generalize the definition of the fixed point for double controlled quasi-metric type spaces with two incomparable functions α and μ which are given as follows.

Theorem 2. Let (G, f) be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu: G \times G \longrightarrow [1, +\infty)$ and let $T: G \longrightarrow G$ be a given mapping. Suppose that the following conditions are satisfied.

There exists $k \in (0, 1)$ such that

$$f(T\dot{c}, Ty) \le k f(\dot{c}, y), \quad \text{for all } \dot{c}, y \in G.$$
(7)

For $v_0 \in G$, choose $v_p = T^p v_0$. Assume that

$$\lim_{i,m \to +\infty} \frac{\alpha(v_{i+1}, v_{i+2})}{\alpha(v_i, v_{i+1})} \mu(v_{i+1}, v_m) < \frac{1}{k}.$$
 (8)

In addition, for every $v \in G$, we have

$$\lim_{p \to +\infty} \alpha(v, v_p) \text{ and } \lim_{p \to +\infty} \mu(v_p, v) \text{ exist and are finite.}$$
(9)

Then, T has a unique fixed point $v^* \in G$.

Proof. Let $v_0 \in G$ be an arbitrary element and $\{v_p\}$ be the sequence defined as above. If $v_0 = Tv_0$, then v_0 be a fixed point of *T*. By (7), we have

$$f(v_p, v_{p+1}) \le k^p f(v_0, v_1), \quad p \in \mathbb{N}.$$

$$(10)$$

For all natural numbers p < m, we have

$$\begin{aligned} f(v_{p}, v_{m}) &\leq \alpha(v_{p}, v_{p+1}) f(v_{p}, v_{p+1}) + \mu(v_{p+1}, v_{m}) f(v_{p+1}, v_{m}) \\ &\leq \alpha(v_{p}, v_{p+1}) f(v_{p}, v_{p+1}) + \mu(v_{p+1}, v_{m}) \alpha(v_{p+1}, v_{p+2}) f(v_{p+1}, v_{p+2}) \\ &+ \mu(v_{p+1}, v_{m}) \mu(v_{p+2}, v_{m}) f(v_{p+2}, v_{m}) \\ &\leq \alpha(v_{p}, v_{p+1}) f(v_{p}, v_{p+1}) + \mu(v_{p+1}, v_{m}) \alpha(v_{p+1}, v_{p+2}) f(v_{p+1}, v_{p+2}) \\ &+ \mu(v_{p+1}, v_{m}) \mu(v_{p+2}, v_{m}) \alpha(v_{p+2}, v_{p+3}) f(v_{p+3}, v_{m}) \\ &\leq \cdots \\ &\leq \alpha(v_{p}, v_{p+1}) f(v_{p}, v_{p+1}) + \sum_{i=p+1}^{m-2} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) f(v_{i}, v_{i+1}) + \prod_{k=p+1}^{m-1} \mu(v_{k}, v_{m}) f(v_{m-1}, v_{m}) \end{aligned}$$
(11)
$$&\leq \alpha(v_{p}, v_{p+1}) k^{p} f(v_{0}, v_{1}) + \sum_{i=p+1}^{m-2} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) k^{i} f(v_{0}, v_{1}) + \prod_{i=p+1}^{m-1} \mu(v_{i}, v_{m}) k^{m-1} f(v_{0}, v_{1}) \\ &\leq \alpha(v_{p}, v_{p+1}) k^{p} f(v_{0}, v_{1}) + \sum_{i=p+1}^{m-1} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) k^{i} f(v_{0}, v_{1}) \\ &\leq \alpha(v_{p}, v_{p+1}) k^{p} f(v_{0}, v_{1}) + \sum_{i=p+1}^{m-1} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) k^{i} f(v_{0}, v_{1}) \\ &\leq \alpha(v_{p}, v_{p+1}) k^{p} f(v_{0}, v_{1}) + \sum_{i=p+1}^{m-1} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) k^{i} f(v_{0}, v_{1}) \\ &\leq \alpha(v_{p}, v_{p+1}) k^{p} f(v_{0}, v_{1}) + \sum_{i=p+1}^{m-1} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) k^{i} f(v_{0}, v_{1}) . \end{aligned}$$

Let

$$S_{p} = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \mu(v_{j}, v_{m}) \right) \alpha(v_{i}, v_{i+1}) k^{i}.$$
(12)

Hence, we have

$$f(v_p, v_m) \le f(v_0, v_1) \left[k^p \alpha (v_p, v_{p+1}) + S_{m-1} - S_p \right].$$
(13)

Let $a_i = (\prod_{j=0}^i \mu(v_j, v_m))\alpha(v_i, v_{i+1})k^i$. By using (8), we have $\lim_{i \to +\infty} (a_{i+1}/a_i) < 1$. By the ratio test, the infinite series $\sum_{i=1}^{\infty} (\prod_{j=0}^{i} \mu(v_j, v_m)) \alpha(v_i, v_{i+1}) k^i$ is convergent, and let p, m tend to infinity in (13), which implies that

$$\lim_{p,m \to +\infty} f(v_p, v_m) = 0.$$
(14)

Since (G, f) is a left complete double controlled quasimetric type space, there exists some $v^* \in G$ such that

$$\lim_{p \to +\infty} f(v_p, v^*) = \lim_{p \to +\infty} f(v^*, v_p) = 0.$$
(15)

By using (Q2) and (7), we have

$$f(v^{*}, Tv^{*}) \leq \alpha(v^{*}, v_{p+1})f(v^{*}, v_{p+1}) + \mu(v_{p+1}, Tv^{*})f(v_{p+1}, Tv^{*}) \leq \alpha(v^{*}, v_{p+1})f(v^{*}, v_{p+1}) + k\mu(v_{p+1}, Tv^{*})f(v_{p}, v^{*})f(v^{*}, Tv^{*}).$$
(16)

By taking limit p tends to infinity together with (9) and (15), we get $f(v^*, Tv^*) = 0$, that is, $Tv^* = v^*$. Now, we have to show that the fixed point of *T* is unique for this; let $\xi \in G$ be such that $T\xi = \xi$ and $v^* \neq \xi$, so we have

So, $v^* = \xi$. Hence, v^* is a unique fixed point of *T*. \Box

Example 2. Let $G = \{0, 1, 2\}$. Define $f: G \times G \longrightarrow [0, +\infty)$ by

$$f(\dot{c}, y) = 0 = 1 = 2$$

$$0 = 0 = \frac{3}{4} = \frac{1}{8}$$

$$1 = \frac{2}{5} = 0 = \frac{4}{5}$$

$$2 = \frac{1}{5} = \frac{1}{4} = 0$$

$$Given \ \alpha, \mu: G \times G \longrightarrow [1, +\infty) \text{ as}$$

$$\alpha(v, y) = 0 = 1 = 2$$

$$0 = 1 = \frac{21}{20} = 2$$

$$1 = \frac{3}{2} = 1 = 1$$

$$\mu(v, y) = 0 = 1 = 2$$

$$0 = 1 = \frac{11}{10} = \frac{5}{3}$$

$$1 = 1 = 1 = \frac{10}{9}$$

$$2 = 1 = 2 = 1$$

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(17)

It is easy to see that (G, f) is a double controlled quasimetric type and the given function f is not a controlled metric type for the function α . Indeed,

$$f(1,2) = \frac{4}{5} > \frac{63}{80} = \alpha(1,0)f(0,2) + \alpha(0,1)f(0,1).$$
(20)

Take T0 = T2 = 0, T1 = 2, and k = 1/2. We observe the following cases:

(i) If $\dot{c} = 0$ and y = 1, we have

$$f(T\dot{c},Ty) = \frac{1}{8} < \frac{1}{2} \times \frac{3}{4} = kf(\dot{c},y).$$
(21)

So, inequality (7) holds. Also, it holds if $\dot{c} = 1$ and.

- (ii) Inequality (7) holds trivially in the cases when $\dot{c} = 0$ and y = 2 and if $\dot{c} = 2$ and y = 0.
- (iii) If $\dot{c} = 1$ and y = 2, we get

$$f(T\dot{c}, Ty) = \frac{1}{5} < \frac{1}{2} \times \frac{4}{5} = kf(\dot{c}, y).$$
(22)

Similarly, in the case when $\dot{c} = 2$ and y = 1, we have $f(T\dot{c}, Ty) < kf(\dot{c}, y)$. So, (7) holds for all cases. Now, let $v_0 = 2$, so we have $v_1 = Tv_0 = T2 = 0$ and $v_2 = 0$, $v_3 = 0$, ...

$$\lim_{i,m \to +\infty} \frac{\alpha(v_{i+1}, v_{i+2})}{\alpha(v_i, v_{i+1})} \mu(v_{i+1}, v_m) = 1 < 2 = \frac{1}{k}.$$
 (23)

That is, (8) holds. In addition, for each $v \in G$, we have

$$\lim_{p \longrightarrow +\infty} \alpha(v, v_p) < \infty \text{ and } \lim_{p \longrightarrow +\infty} \mu(v_p, v) < \infty.$$
(24)

That is, (9) holds. Hence, all conditions of Theorem 2 are satisfied, and v = 0 is a unique fixed point.

3. Further Results

In this section, we introduce the concept of $\alpha - \mu - k$ double controlled contraction and prove related fixed point results with some examples.

Definition 6. Let G be a nonempty set, (G, f) be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu: G \times G \longrightarrow [1, +\infty)$, and $\check{T}: G \longrightarrow G$ be a given mapping. Assume there exists $k \in (0, 1/2)$ such that

$$h_{1} = \sup\{k\alpha(\dot{c}, y), \dot{c}, y \in G\} < \frac{1}{2},$$

$$h_{2} = \sup\{k\mu(\dot{c}, y), \dot{c}, y \in G\} < \frac{1}{2}.$$
(25)

Suppose that the following conditions are satisfied:

$$f(\check{T}\dot{c},\check{T}y) \le k[f(\dot{c},\check{T}y) + f(y,\check{T}\dot{c})], \quad \text{for all } \dot{c}, y \in G.$$
(26)

For $v_0 \in G$ and $v_p = \check{T}^p v_0$, we have

$$\lim_{i,m \to +\infty} \frac{\alpha(v_{i+1}, v_{i+2})}{\alpha(v_i, v_{i+1})} \mu(v_{i+1}, v_m) < \frac{1-h}{h},$$
(27)

where $h = \max\{h_1, h_2\}$. Also, for each $v \in G$, we have

$$\lim_{\substack{p \longrightarrow +\infty \\ p \longrightarrow +\infty}} \alpha(v_p, v) < \infty,$$

$$\lim_{\substack{p \longrightarrow +\infty \\ p \longrightarrow +\infty}} \mu(v_p, v) < \infty.$$
(28)

Then, \check{T} is called $\alpha - \mu - k$ double controlled contraction.

Theorem 3. Let (G, f) be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu: G \times G$ $\longrightarrow [1, +\infty)$ and let $\check{T}: G \longrightarrow G$ be $\alpha - \mu - k$ double controlled contraction. Then, \check{T} has a unique fixed point $v^* \in G$.

Proof. Let $v_0 \in G$ be an arbitrary element and $\{v_p\}$ be the sequence defined as above. If $v_0 = \check{T}v_0$, then v_0 is a fixed point of \check{T} . By (26), we have

$$f(v_{p}, v_{p+1}) = f(\check{T}v_{p-1}, \check{T}v_{p}) \le k [f(v_{p-1}, \check{T}v_{p}) + f(v_{p}, \check{T}v_{p-1})] \le k [f(v_{p-1}, v_{p+1}) + f(v_{p}, v_{p})] \le k \alpha (v_{p-1}, v_{p}) f(v_{p-1}, v_{p}) + k \mu (v_{p}, v_{p+1}) f(v_{p}, v_{p+1}) f(v_{p}, v_{p+1}) \le h_{1} f(v_{p-1}, v_{p}) + h_{2} f(v_{p}, v_{p+1}), \text{ (by Definition 6)} \le h f(v_{p-1}, v_{p}) + h f(v_{p}, v_{p+1}) (1-h) f(v_{p}, v_{p+1}) \le h f(v_{p-1}, v_{p}), f(v_{p}, v_{p+1}) \le \frac{h}{1-h} f(v_{p-1}, v_{p}).$$
(30)

Complexity

Now,

$$f(v_{p-1}, v_p) = f(\check{T}v_{p-2}, \check{T}v_{p-1}) \le \max\{f(\check{T}v_{p-2}, \check{T}v_{p-1}), f(\check{T}v_{p-1}, \check{T}v_{p-2})\},\$$

$$\le k[f(v_{p-2}, v_p) + f(v_{p-1}, v_{p-1})],\$$

$$\le k\alpha(v_{p-2}, v_{p-1})f(v_{p-2}, v_{p-1}) + k\mu(v_{p-1}, v_p)f(v_{p-1}, v_p),\$$

$$\le h_1 f(v_{p-2}, v_{p-1}) + h_2 f(v_{p-1}, v_p),\$$
(31)

$$(1-h)f(v_{p-1},v_p) \le hf(v_{p-2},v_{p-1}),$$

$$f(v_{p-1},v_p) \le \frac{h}{1-h}f(v_{p-2},v_{p-1})$$

Combining (30) and the above inequality, we get

$$f(v_{p}, v_{p+1}) \leq \left(\frac{h}{1-h}\right)^{2} f(v_{p-2}, v_{p-1}).$$
(32)

Continuing in this way, we obtain

$$f(v_p, v_{p+1}) \le \left(\frac{h}{1-h}\right)^p f(v_0, v_1).$$
 (33)

Now, to prove that $\left\{ v_p \right\}$ is a Cauchy sequence, for all natural numbers p < m, we have

$$f(v_{p}, v_{m}) \leq \alpha(v_{p}, v_{p+1})f(v_{p}, v_{p+1}) + \sum_{i=p+1}^{m-2} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m})\right) \alpha(v_{i}, v_{i+1})f(v_{i}, v_{i+1}) + \prod_{k=p+1}^{m-1} \mu(v_{k}, v_{m})f(v_{m-1}, v_{m}).$$
(34)

Using (33), we get

$$f(v_{p}, v_{m}) \leq \alpha(v_{p}, v_{p+1}) \left(\frac{h}{1-h}\right)^{p} f(v_{0}, v_{1}) \\ + \sum_{i=p+1}^{m-2} \left(\prod_{j=p+1}^{i} \mu(v_{j}, v_{m})\right) \alpha(v_{i}, v_{i+1}) \left(\frac{h}{1-h}\right)^{i} f(v_{0}, v_{1}) \\ + \prod_{i=p+1}^{m-1} \mu(v_{i}, v_{m}) \left(\frac{h}{1-h}\right)^{m-1} f(v_{0}, v_{1}) \\ \leq \alpha(v_{p}, v_{p+1}) \left(\frac{h}{1-h}\right)^{p} f(v_{0}, v_{1}) \\ + \sum_{i=p+1}^{m-1} \left(\prod_{j=0}^{i} \mu(v_{j}, v_{m})\right) \alpha(v_{i}, v_{i+1}) \left(\frac{h}{1-h}\right)^{i} f(v_{0}, v_{1}), \\ S_{p} = \sum_{i=0}^{p} \left(\prod_{j=0}^{i} \mu(v_{j}, v_{m})\right) \alpha(v_{i}, v_{i+1}) \left(\frac{h}{1-h}\right)^{i}.$$

$$(35)$$

Hence, we have

$$f(v_{p}, v_{m}) \leq f(v_{0}, v_{1}) \left[\left(\frac{h}{1-h} \right)^{p} \alpha \left(v_{p}, v_{p+1} \right) + S_{m-1} - S_{p} \right].$$
(36)

Let $a_i = (\prod_{j=0}^i \mu(v_j, v_m))\alpha(v_i, v_{i+1})(h/(1-h))^i$. By using (27), we have $\lim_{i \to +\infty} a_{i+1}/a_i < 1$. By the ratio test, the

infinite series $\sum_{i=1}^{\infty} (\prod_{j=0}^{i} \mu(v_j, v_m)) \alpha(v_i, v_{i+1}) (h/(1-h))^i$ is convergent, and let p, m tend to infinity in (36), which yield

$$\lim_{p,m\longrightarrow+\infty} f(v_p, v_m) = 0.$$
(37)

So, the sequence $\{v_p\}$ is a left Cauchy. Since (G, f) is a left complete double controlled quasi-metric type space, there must be exist some $v^* \in G$ such that

$$\lim_{p \to +\infty} f(v_p, v^*) = 0 = \lim_{p \to +\infty} f(v^*, v_p).$$
(38)

$$\begin{aligned} f(v^*, \check{T}v^*) &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) f(v_{p+1}, \check{T}v^*) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k[f(v_p, \check{T}v^*) + f(v^*, v_{p+1})] \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 [f(v_{p-1}, \check{T}v^*) + f(v^*, v_p)] + \mu(v_{p+1}, \check{T}v^*) kf(v^*, v_{p+1}) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 [\alpha(v_{p-1}, v^*) f(v_{p-1}, v^*) \\ &+ \mu(v^*, \check{T}v^*) f(v^*, \check{T}v^*) + f(v^*, v_p)] + \mu(v_{p+1}, \check{T}v^*) kf(v^*, v_{p+1}) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 (\alpha(v_{p-1}, v^*) f(v_{p-1}, v^*) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 [f(v_{p-1}, \check{T}v^*) + f(v^*, v_{p+1})] \end{aligned}$$
(39)
$$&\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 [\alpha(v_{p-1}, v^*) + f(v^*, v_p)] + \mu(v_{p+1}, \check{T}v^*) kf(v^*, v_{p+1}) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 [\alpha(v_{p-1}, v^*) + f(v^*, v_{p+1})] \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, \check{T}v^*) + f(v^*, v_p)] + \mu(v_{p+1}, \check{T}v^*) f(v^*, v_{p+1}) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, \check{T}v^*) + f(v^*, v_p)] + \mu(v_{p+1}, \check{T}v^*) kf(v^*, v_{p+1}) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, \check{T}v^*) + f(v^*, v_p) + \mu(v_{p+1}, \check{T}v^*) kf(v^*, v_{p+1}) \\ &\leq \alpha(v^*, v_{p+1}) f(v^*, v_{p+1}) + \mu(v_{p+1}, \check{T}v^*) k^2 (v^*, v_p) + \mu(v_{p+1}, \check{T}v^*) kf(v^*, v_{p+1}) . \end{aligned}$$

By taking limit as p tend to infinity together with (38), we get

$$(1-(h_2)^2)f(v^*,\check{T}v^*) \le 0.$$
 (40)

Hence, $v^* = \check{T}v^*$, which is a contradiction. Now, we have to show that the fixed point of \check{T} is unique for this let $v^{**} \in G$ such that $\check{T}v^{**} = v^{**}$, so we have

$$f(v^{*}, v^{**}) = f(\check{T}v^{*}, \check{T}v^{**}) \le k [f(v^{*}, \check{T}v^{**}) + f(v^{**}, \check{T}v^{*})]$$

$$\le k [f(v^{*}, v^{**}) + f(v^{**}, v^{*})]$$

$$\le \left(\frac{k}{1-k}\right)^{2} f(v^{**}, v^{*})$$

$$\vdots$$

$$\le \left(\frac{k}{1-k}\right)^{2n} f(v^{*}, v^{**}).$$
(41)

By taking limit as *n* tend to infinity, we have $v^* = v^{**}$. Hence, v^* is a unique fixed point of \check{T} .

Example 3. Take $G = \{0, 1, 2\}$. Define $f: G \times G \longrightarrow [0, +\infty)$ by

$$f(\dot{c}, y) = 0 = 1 = 2$$

$$0 = 0 = \frac{2}{5} = \frac{1}{4}$$

$$1 = -\frac{9}{20} = 0 = -\frac{4}{5}$$
(42)

controlled quasi-metric type space. The given f is not a controlled metric for the function α . Indeed,

$$f(2,1) = \frac{7}{10} > \frac{307}{500} = \alpha(2,0)f(2,0) + \alpha(0,1)f(0,1).$$
(43)

Take $\check{T}0 = \check{T}2 = 2$, $\check{T}1 = 0$, and k = 2/5, and we observe the follows cases:

(i) If $\dot{c} = 0$ and y = 1, we have

$$f(\check{T}\dot{c},\check{T}y) = \frac{1}{5} \le \frac{8}{25} = \frac{2}{5} \left[0 + \frac{4}{5} \right]$$

= $k [f(\dot{c},\check{T}y) + f(y,\check{T}\dot{c})].$ (44)

If $\dot{c} = 1$ and y = 0, we get

$$f(\check{T}\dot{c},\check{T}y) = \frac{1}{4} \le \frac{8}{25} = k[f(\dot{c},\check{T}y) + f(y,\check{T}\dot{c})].$$
(45)

- (ii) It is straightforward in the case when we take $\dot{c} = 0$ and y = 2.
- (iii) If $\dot{c} = 1$ and y = 2, we get

$$\frac{1}{4} \le \frac{2}{5} = \frac{2}{5} \left[\frac{4}{5} + \frac{1}{5} \right] = k [f(\dot{c}, \check{T}y) + f(y, \check{T}\dot{c})].$$
(46)

Similarly, in the case when we take $\dot{c} = 2$ and y = 1, that is, inequality (26) holds, we have

$$h_{1} = \sup\{k\alpha(\dot{c}, y), \dot{c}, y \in G\} < \frac{1}{2},$$

$$h_{2} = \sup\{k\mu(\dot{c}, y), \dot{c}, y \in G\} < \frac{1}{2},$$
(47)

and $h = \max\{h_1, h_2\} = 12/25$. Now, let $v_0 = 1$, and we have $v_1 = \check{T}v_0 = \check{T}1 = 0$, $v_2 = \check{T}v_1 = \check{T}0 = 2$, $v_3 = \check{T}2 = 2$, $v_3 = 2, \ldots$

$$\lim_{i,m \to +\infty} \frac{\alpha(v_{i+1}, v_{i+2})}{\alpha(v_i, v_{i+1})} \mu(v_{i+1}, v_m) = 1 < \frac{1-h}{h},$$
(48)

which shows that (27) holds. In addition, for each $v \in G$, we have

$$\lim_{p \to +\infty} \alpha(v_p, v) < \infty,$$

$$\lim_{p \to +\infty} \alpha(v, v_p) < \infty,$$

$$\lim_{p \to +\infty} \mu(v_p, v) < \infty.$$
(49)

That is, (28) holds. All conditions of Theorem 3 are proved, and v = 2 is the unique fixed point.

Definition 7. Let (G, f) be a complete quasi-*b*-metric space. $\check{T}: G \longrightarrow G$ is called Chatterjee-type *b*-contraction if the following conditions are satisfied:

$$f(\check{T}\dot{c},\check{T}y) \le k[f(\dot{c},\check{T}y) + f(y,\check{T}\dot{c})], \tag{50}$$

for all \dot{c} , $y \in G$, $k \in (0, 1/2)$, and

$$b < \frac{1 - kb}{kb}.$$
 (51)

Theorem 4. Let (G, f) be a complete quasi-b-metric space and $\check{T}: G \longrightarrow G$ be Chatterjee-type b-contraction. Then, \check{T} has a unique fixed point.

Remark 2. In the Example 3, f is a quasi-*b*-metric with $b \ge 16/13$, but we cannot apply Theorem 4 because \check{T} is not a Chatterjee-type *b*-contraction. Indeed, $b \not\le (1 - kb)/kb$, for all $b \ge 16/13$.

4. Conclusion

In the present paper, we have obtained sufficient conditions to ensure the existence of the fixed point for different types of contractive mappings in the setting of double controlled quasi-metric type spaces. Examples are given to demonstrate the variety of our results. New results in quasi-*b*-metric spaces, extended *b*-metric spaces, extended quasi-*b*-metric spaces, controlled metric spaces, and controlled quasimetric spaces can be obtained as corollaries of our results. Also, results in right complete and dual complete double controlled quasi-metric type spaces can be obtained in a similar way. It is natural to ask, Are there other multivalued contraction mappings which can be applied to obtain more results in the double controlled quasi-metric type spaces? Is there interest to find serious applications to integral equations and dynamical systems?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Each author equally contributed to this paper, read, and approved the final manuscript.

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