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Numerical simulation of advection–diffusion equation with caputo-fabrizio time fractional derivative in cylindrical domains: Applications of pseudo-spectral collocation method

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Abstract The study of the unsteady fractional advection–diffusion equation (ADE) is carried out in cylindrical geometry along with time-exponential concentration on a cylindrical surface. We have used the Caputo-Fabrizio time-fractional derivative for the fractional model of the advection–diffusion. The analytical solutions for the solute concentration are determined by using integral transformations. For comparison, we also present a numerical scheme to get a numerical solution for different parameters' values. For time ordinary derivative approximation, we use the finite difference method. For space derivatives, we use a pseudo-spectral collocation method for higher-order accuracies. The advection-diffusion' classical model is obtained by taking the fractional parameter to be unit. The influence of the memory, namely, the Caputo-Fabrizio time-fractional derivative on the solute concentration is studied and compared with the ordinary case. Also, the impact of the drift velocity is analyzed by employing the Peclet number. It has been found that the concentration is decreasing with the Peclet number and is increasing with the radial coordinate. The present investigation will be helpful in future research to use a higher-order approximation for

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ordinary derivatives. As the derivatives in space are ordinary derivatives, we use highly accurate pseudo-spectral collocation approximation for them.

1. Introduction

In various areas of physics, biology, and engineering, fractional calculus has achieved a mileage stone in applications of the areas as mentioned earlier [1–5]. The diffusion or heat conduction, along with the velocity field, transport process, or ground hydrology, is well explained by the time-fractional (ADE) advection–diffusion equation. This equation can be obtained as a result of the balanced equation for the time-nonlocal constitutive equation. Some problems modeled by the advection–diffusion are transport phenomena in food processing, groundwater solute transport, atmospheric pollutant transport, wellbores flows, porous soil solute diffusion, geothermal production with reinjection, and energy storage in porous media. The classical differential equation did not describe well many diffusion phenomena like super diffusion and slow diffusion. The time-fractional derivatives models describe such processes and their physical aspects very efficiently. Povstenko [6] has provided an excellent comparison of the sub-diffusion regime and the supper-diffusion regime with ordinary diffusion. Various solution methods have been used to solve the (ADE) advection–diffusion equation due to its extensive application in different fields. Analytical solution to fractional (ADE) advection–diffusion equation with time-fractional pulses on the boundary has been found by Rubbab et al. [7]. A time-fractional heat conduction's solution with physical Robin condition on the sphere surface in a two-layered slab, and the perfect contact of layers has been presented by Kukla et al. [8]. Massabo et al. [9] have discussed the analytical solution by comparing it with the famous two-dimensional infinite domain with some limiting analytical solutions. Peclet numbers have been used to examine the influence of the boundary conditions on solutions, and the solution is compared with experimental data. The generalized analytical solution for ADE (advection–diffusion equation) in the finite space domain having an arbitrary time-dependent inlet boundary condition has been derived by Chen et al. [10]. An analytical solution of the convective-diffusion in two dimensions in a single fiber isotropic capillary membrane for solute transportation has been presented by Godongwana et al. [11].

Another essential analytical solution for the convection–diffusion problem has been discussed by Ivanchenko et al. [12]. The use of local fractional Laplace equation in solving fractional nonlinear CDE's (convection–diffusion equations) with improved Riemann-Liouville derivative has been studied by Merdan et al. [13]. The discretization method based on unstructured grids using a finite volume approach is proposed to solve CDE's (convection–diffusion equations) in r-z coordinates. A perfect control volume calculation method is presented by Yu et al. [14].

In this paper, the time-fractional differential equation is used for the advection–diffusion process in cylindrical geometry. Caputo and Fabrizio's time-fractional derivative is introduced in formulating the governing equation. A new analytical solution of the ADE (advection–diffusion equation) in a cylindrical coordinate with initial and time-exponential boundary source is obtained by employing the integral transforms. The ordinary ADE (advection–diffusion equation) solution is a particular case of our solution by taking the limiting value of the fractional parameter of the time-derivative to be unit. The numerical calculations and graphical illustrations are used to present the fractional parameter (the memory effect, i.e., the effects on the flow with history) and the drift velocity (*Pe*, the Peclet number) on the concentration. To find the solution to a complex problem, the finite element method [15–21] and the finite difference method are famous. Numerical schemes are used to solve general-purpose problems. When we have to deal with fraction-order derivatives, the classical approach is not applicable for the discretization of fractional-order derivatives. The reason is apparent due to the global effect of fractional-order derivatives. Usually, fractional-order derivatives are presented in integrals, and ordinary derivatives are inside of them. In a numerical strategy, we first approximate the integral that gives us a linear combination of ordinary derivatives. Once we have an expression that involves ordinary derivatives, we may use a classical method to approximate ordinary derivatives. In the present research, we use the first-order approximation of ordinary derivatives using the forward difference method. It is of interest in future research to use a higher-order approximation for ordinary derivatives. As the derivatives in space are ordinary derivatives, we use highly accurate pseudo-spectral collocation approximation for them.

The paper is organized in the following manner. Section 1 has added the introduction based on the essential investigations on fractional derivative and numerical techniques. The scope of this investigation is also part of this section. Section 2 contains the mathematical formulation and analytical solution of the developed mathematical model. A brief introduction of the numerical technique, namely, the pseudo-spectral collocation method, is added in Section 3. A comparison between analytical and numerical solution is added in Section 4. The results and discussion of the numerical solution are highlighted in Section 5. This study's conclusions are added in the second last section, and the essential references related to this work are cited in the previous section of this paper.

2. Formulation of the problem and solution

The one-dimensional unsteady advection–diffusion equation with constant diffusivity $D > 0$, in the cylindrical domain with radius > 0, is given as, [12,14]

$$\frac{\partial \psi'(r',t')}{\partial t'} + \frac{V_0}{r'}\frac{\partial \psi'(r',t')}{\partial r'} = D\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial \psi'(r',t')}{\partial r'}\right), \quad r' \in [0,R],\ t' \geqslant 0, \tag{1}$$

where $\psi'(r',t')$represents the species concentration and V_0 the constant transport parameter.

Along with the advection–diffusion Eq. (1), with following initial and boundary conditions:

$$\psi'(r',0)=\psi_0\left(\frac{r'}{R}\right)^{V_0/D}, \tag{2}$$

$$\psi'(0,t')=0,\ \ \psi'(R,t')=\psi_0\exp(-k_0t'),\quad k_0\geqslant 0, \tag{3}$$

where $\psi_0$is the constant species concentration. Consider the following dimensionless quantities

$$r=\frac{r'}{R},\quad t=\frac{Dt'}{R^2},\quad \psi=\frac{\psi'}{\psi_0},\quad Pe=\frac{V_0}{D}, \tag{4}$$

Eqs. (1)–(3) take the form

$$\frac{\partial\psi(r,t)}{\partial t}+Pe\frac{1}{r}\frac{\partial\psi(r,t)}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r,t)}{\partial r}\right),\quad r\in[0,1],\ t\geqslant 0, \tag{5}$$

$$\psi(r,0)=F(r)=r^{Pe}, \tag{6}$$

$$\psi(0,t)=0,\ \psi(1,t)=\exp(-a_0t),\quad a_0=\frac{k_0R^2}{D}. \tag{7}$$

In the above relations, the Peclet number, namely, the advective and diffusive transport ratio. We introduce the following transformation in Eqs. (5)–(7)

$$\psi(r,t)=r^p\vartheta(r,t),\quad p=Pe/2, \tag{8}$$

we get

$$\left(\frac{p^2}{r^2}+\frac{\partial}{\partial t}\right)\vartheta(r,t)=\frac{\partial^2\vartheta(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta(r,t)}{\partial r}, \tag{9}$$

$$\vartheta(r,0)=r^{-p}F(r)=G(r)=r^p, \tag{10}$$

$$\vartheta(0,t)=0,\ \vartheta(1,t)=\exp(-a_0t). \tag{11}$$

The fractional form of the unsteady advection–diffusion Eq. (9) with CF (Caputo-Fabrizio) time-fractional derivative operator ${}^{CF}D_t^{\alpha}$ can be written as

$${}^{CF}D_t^{\alpha}\vartheta(r,t)=\frac{\partial^2\vartheta(r,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta(r,t)}{\partial r}-\frac{p^2}{r^2}\vartheta(r,t), \tag{12}$$

where the CF (Caputo-Fabrizio) time-fractional derivative operator ${}^{CF}D_t^{\alpha}$ is given by [22–23]

$${}^{CF}D_t^{\alpha}\vartheta(r,t)=\frac{1}{1-\alpha}\int_0^t\frac{\partial\vartheta(r,\tau)}{\partial\tau}\exp\left(\frac{-\alpha(t-\tau)}{1-\alpha}\right)d\tau,\quad 0<\alpha<1. \tag{13}$$

with the properties [6,18,19]:

$$L\left\{\ {}^{CF}D_t^{\alpha}\vartheta(r,t)\right\}=\frac{qL\{\vartheta(r,t)\}-\vartheta(r,0)}{(1-\alpha)q+\alpha}, \tag{14}$$

$$\lim_{\alpha\to 1}{}^{CF}D_t^{\alpha}\vartheta(r,t)=\frac{\partial\vartheta(r,t)}{\partial t}, \tag{15}$$

where $L\{\vartheta(r,t)\}=\bar{\vartheta}(r,q)=\int_0^{\infty}\vartheta(r,t)e^{-qt}dt$ is the Laplace transform of a function $\vartheta(r,t)$. Eq. (13) is known as the first-order time discretization of the updated developed fractional-order derivative. The information related to this operator can be seen from these Refs. [24,25].

2.1. Analytical solution for the time-fractional advection–diffusion process

To find an analytical solution for the fractional diffusion Eq. (12) along with the initial and boundary conditions (10) and (11), we will use Laplace transformation coupled with the finite Hankel transform of order p.

Applying the Laplace transform to Eqs. (11) and (12), using Eqs. (10) and (14), we obtain the following transformed form:

$$\frac{bq}{q+\alpha b}\bar{\vartheta}(r,q)-\frac{b}{q+\alpha b}G(r)=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{p^2}{r^2}\right)\bar{\vartheta}(r,q), \tag{16}$$

$$\bar{\vartheta}(0,q)=0,\quad \bar{\vartheta}(1,q)=\frac{1}{q+a_0}, \tag{17}$$

where $b=\frac{1}{1-\alpha}$, $0<\alpha<1$.

Now, applying the finite Hankel transform [26] of order $p$to Eq. (16) and by using Bessel functions' properties, we have

$$\frac{bq}{q+\alpha b}\tilde{\vartheta}_p(s_n,q)-\frac{b}{q+\alpha b}\frac{J_{p+1}(s_n)}{s_n}=s_nJ_{p+1}(s_n)\bar{\vartheta}(1,q)-s_n^2\tilde{\vartheta}_p(s_n,q), \tag{18}$$

where s_n, $n=1,2,\ldots$ are the + ve roots of the transcendental equation $J_p(x)=0$and $\tilde{\vartheta}_p(s_n,q)=\int_0^1\bar{\vartheta}(r,q)rJ_p(rs_n)dr$ the finite Hankel transform of the function $\bar{\vartheta}(r,q)$.

Using Eqs. (17) and (18) we obtain

$$\tilde{\vartheta}_p(s_n,q)=\frac{s_nJ_{p+1}(s_n)}{q+a_0}\frac{q+\alpha b}{(b+s_n^2)q+\alpha bs_n^2}+\frac{b}{(b+s_n^2)q+\alpha bs_n^2}\frac{J_{p+1}(s_n)}{s_n}. \tag{19}$$

Eq. (19) can be simplified as

$$\tilde{\vartheta}_p(s_n,q)=\frac{J_{p+1}(s_n)}{s_n(q+a_0)}+\frac{a_0bJ_{p+1}(s_n)}{s_n[(\alpha b-a_0)s_n^2-a_0b](q+a_0)}-\frac{\alpha b^2s_nJ_{p+1}(s_n)}{(b+s_n^2)[(\alpha b-a_0)s_n^2-a_0b]}\frac{1}{q+\frac{\alpha bs_n^2}{b+s_n^2}}+\frac{b}{(b+s_n^2)}\frac{J_{p+1}(s_n)}{s_n}\frac{1}{q+\frac{\alpha bs_n^2}{b+s_n^2}}. \tag{20}$$

Firstly, by applying inverse Laplace transform, inverse Hankel transform, and by using the integral [18] $\int_0^1 r^{p+1}J_p(rs_n)dr=\frac{J_{p+1}(s_n)}{s_n}$, we have

$$\vartheta(r,t)=r^pe^{-a_0t}+2a_0be^{-a_0t}\sum_{n=1}^{\infty}\frac{J_p(rs_n)}{s_n[(\alpha b-a_0)s_n^2-a_0b]J_{p+1}(s_n)}-2a_0b\sum_{n=1}^{\infty}\frac{J_p(rs_n)}{s_n[(\alpha b-a_0)s_n^2-a_0b]J_{p+1}(s_n)}\exp\left(-\frac{\alpha bs_n^2t}{b+s_n^2}\right). \tag{21}$$

By using Eq. (8), we get the species concentration corresponding to the time-fractional advection–diffusion process, as

$$\psi(r,t)=r^p\vartheta(r,t), \tag{22}$$

where $\vartheta(r,t)$is given by Eq. (21).

2.2. Analytical solution for the ordinary advection–diffusion process ($\alpha \to 1$, $b \to \infty$)

To find the solutions corresponding to the ordinary advection–diffusion process, we apply the limits $\alpha \to 1$, $b \to \infty$ in Eq. (21)

$$\vartheta_c(r,t) = r^p e^{-a_0 t} + 2a_0 \sum_{n=1}^{\infty} \times \frac{J_p(rs_n)}{s_n(s_n^2 - a_0)J_{p+1}(s_n)} \left[\exp(-a_0 t) - \exp(-s_n^2 t)\right]. \quad (23)$$

where $\vartheta_c(r,t) = \lim_{\substack{\alpha \to 1,\\ b \to \infty}} \vartheta(r,t)$. The corresponding species concentration is given by

$$\psi_c(r,t) = r^p \vartheta_c(r,t). \quad (24)$$

3. Numerical technique

The numerical discretization of fractional order derivatives is challenging compared to the discretization of ordinary derivatives. We define initial $t_1 = 0$ and final time t_n. We take uniform step-size Δt and n_t the total number of t grid-points. The time partition can be written as

$$\{t_1, t_2, t_3, \cdots, t_n\}$$

Caputo-Fabrizio time-fraction derivative operation can be written as

$$CFD_t^{\alpha}\vartheta(r,t_i) = \frac{1}{1-\alpha}\int_{t_j}^{t_i} \exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)\frac{\partial\vartheta(t_i,\tau)}{\partial\tau}d\tau. \quad (25)$$

The integral can be subdivided into small intervals as

$$CFD_t^{\alpha}\vartheta(r,t_i) = \frac{1}{1-\alpha}\sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}} \times \exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)\frac{\partial\vartheta(t_i,\tau)}{\partial\tau}d\tau. \quad (26)$$

We can approximate the integral

$$\int_{t_j}^{t_{j+1}} \exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)\frac{\partial\vartheta(t_i,\tau)}{\partial\tau}d\tau \approx \frac{\partial\vartheta(r,\tau)}{\partial\tau}\Big|_{\tau=t_{j+\frac{1}{2}}}\int_{t_j}^{t_{j+1}} \exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)d\tau. \quad (27)$$

Furthermore, we can compute the integral

$$\int_{t_j}^{t_{j+1}} \exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)d\tau,$$

analytical, and it has the value

$$\frac{1}{1-\alpha}\int_{t_j}^{t_{j+1}} \exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)d\tau = \frac{1}{\alpha}\left(1-\exp\left(\frac{-\alpha(t_i-\tau)}{1-\alpha}\right)\right)\exp\left(\frac{\alpha\Delta t}{1-\alpha}(j+1-i)\right). \quad (28)$$

And

$$g(j+1-i,\Delta t,\alpha) = \exp\left(\frac{\alpha\Delta t}{1-\alpha}(j+1-i)\right).$$

For $i = n_t$ and $j = 1, 2, \cdots .., n_t - 1$,we have the vector

$$\boldsymbol{g} = [g(0), g(-1), g(-2), \cdots, g(2-nt]^T. \quad (29)$$

We can construct a matrix of size $(n_t - 1) \times (n_t - 1)$ form of the vector $\boldsymbol{g}$ as

$$A = \begin{bmatrix} g(0) & & & & \\ g(-1) & g(0) & & & \\ g(-2) & g(-1) & g(0) & & \\ \vdots & \vdots & & & \\ g(2-n_t) & g(2-n_t) & \cdots & g(-1) & g(0) \end{bmatrix} \quad (30)$$

Approximation of time-fraction derivative can be written as

$$\begin{bmatrix} {}^{CF}D_t^{\alpha}\vartheta(r,t_1) \\ {}^{CF}D_t^{\alpha}\vartheta(r,t_2) \\ \cdot \\ \vdots \\ {}^{CF}D_t^{\alpha}\vartheta(r,t_n) \end{bmatrix} \approx \begin{bmatrix} g(0) & & & \\ g(-1) & g(0) & & \\ g(-2) & g(-1) & g(0) & \\ \vdots & \vdots & & \\ g(2-n_t) & g(3-n_t)\cdots & g(-1) & g(0) \end{bmatrix} \begin{bmatrix} \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{1+\frac{1}{2}}} \\ \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{2+\frac{1}{2}}} \\ \vdots \\ \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{n_t-1+\frac{1}{2}}} \end{bmatrix} \quad (31)$$

The approximation of vector

$$\begin{bmatrix} \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{1+\frac{1}{2}}} \\ \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{2+\frac{1}{2}}} \\ \vdots \\ \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{n_t-1+\frac{1}{2}}} \end{bmatrix}$$

is computed as

$$\begin{bmatrix} \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{1+\frac{1}{2}}} \\ \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{2+\frac{1}{2}}} \\ \vdots \\ \frac{\partial\vartheta(r,\tau)}{\partial\tau}|_{\tau=t_{n_t-1+\frac{1}{2}}} \end{bmatrix} \approx \frac{1}{\Delta t}\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} \vartheta_2(r) \\ \vartheta_3(r) \\ \vdots \\ \vartheta_{n_t}(r) \end{bmatrix} 23em - \frac{\vartheta_1(r)}{\Delta t}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (32)$$

where $\theta_i(r) = \theta(t_i, r)$.By using the above approximation and (31), we get

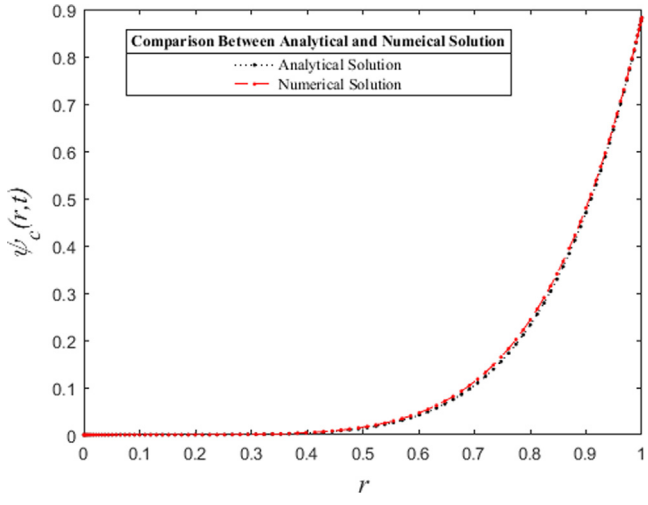

Fig. 1 Comparison between analytical and numerical solutions.

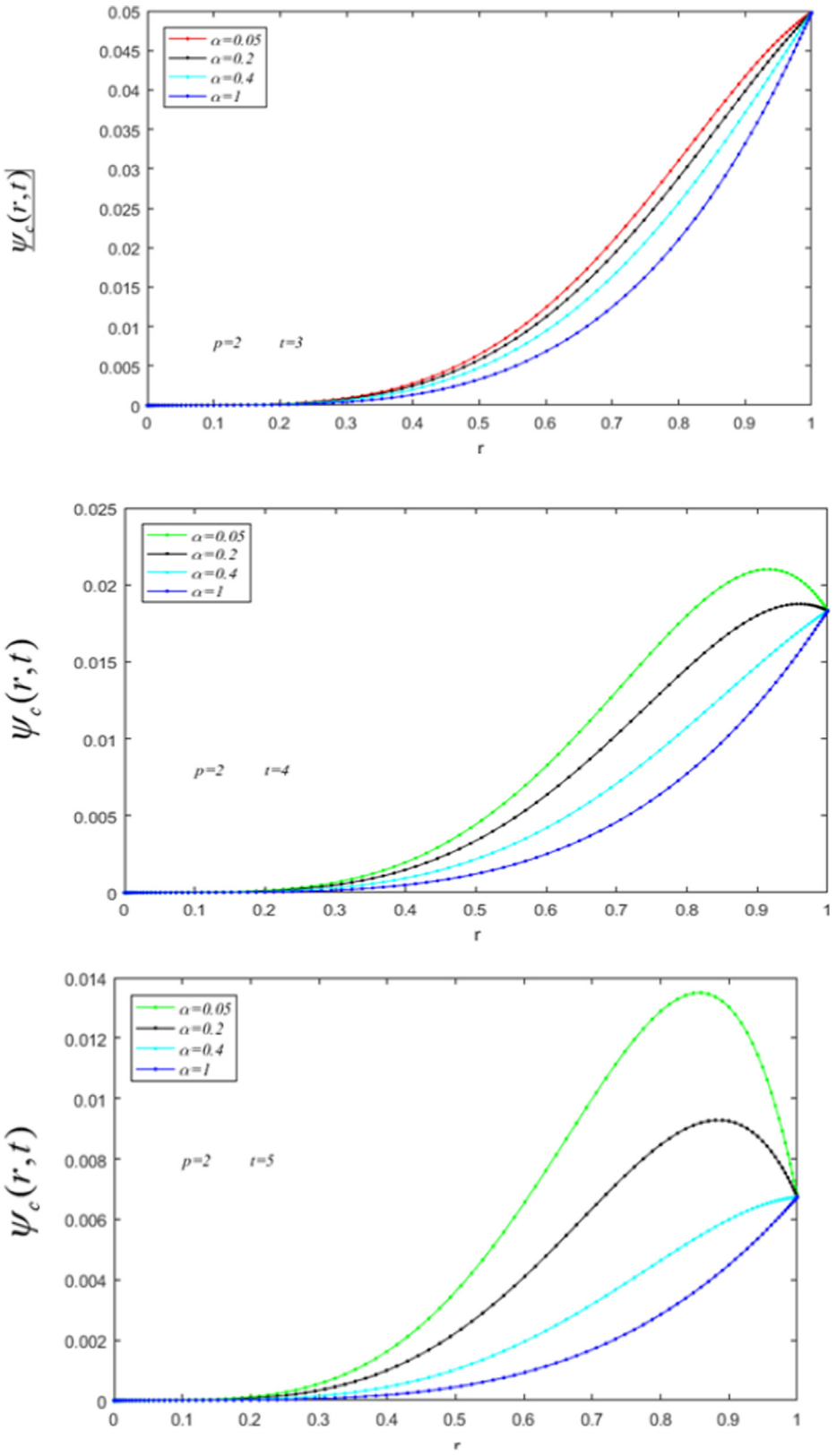

Fig. 2 The dynamic evolution of the concentration $\psi_c(r, t)$ for the different fractional parameters α different values of the time t, $a_0 = 1$, and Peclet number $Pe = 4$.

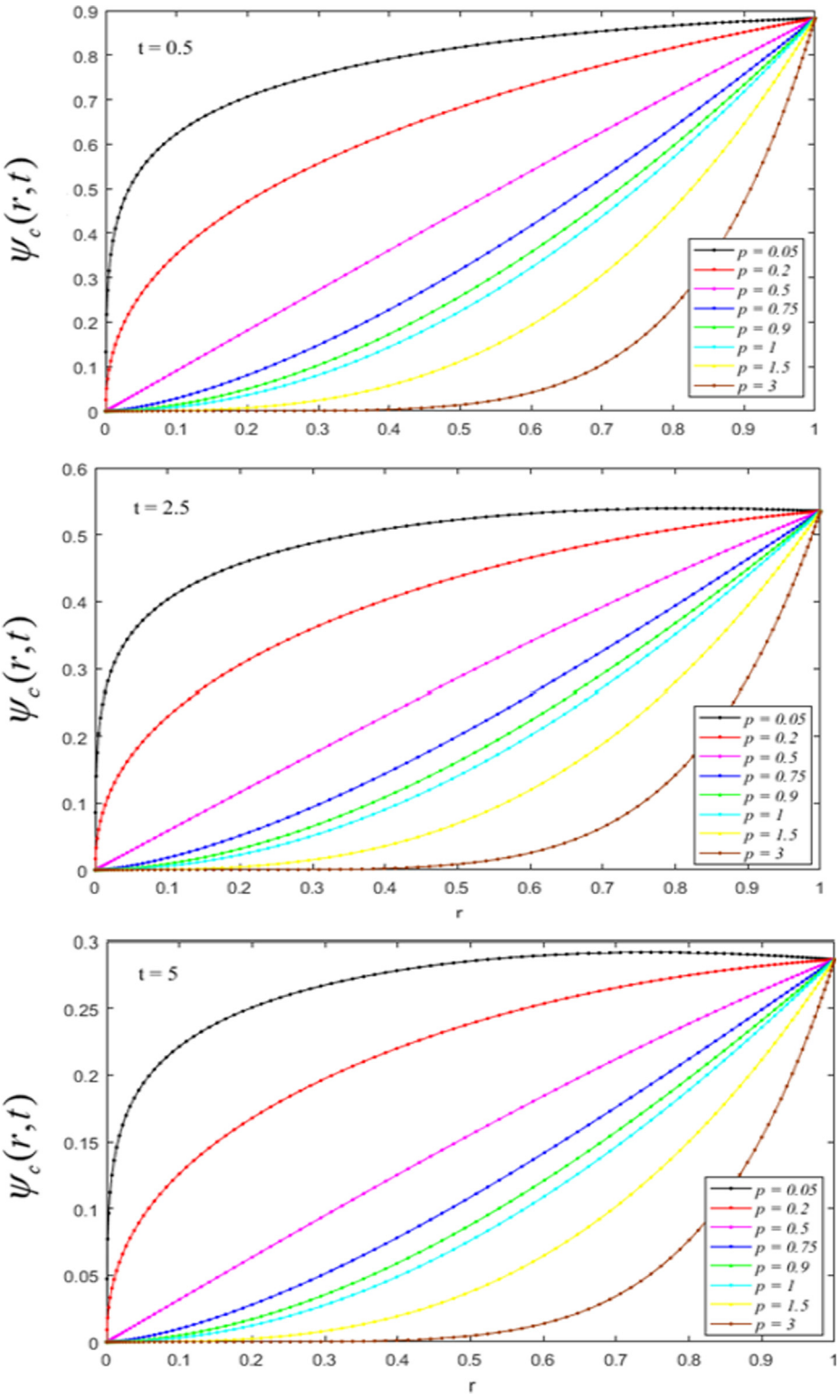

Fig. 3 The dynamic evolution of the concentration $\psi_c(r, t)$for a range of the Peclet number, different time t, $a_0 = 0.25$, and the fractional parameter $\alpha = 0.5$.

$$\begin{bmatrix} {}^{CF}D_i^{\alpha}\vartheta(r,t_1) \\ {}^{CF}D_i^{\alpha}\vartheta(r,t_2) \\ \cdot \\ \vdots \\ {}^{CF}D_i^{\alpha}\vartheta(r,t_n) \end{bmatrix} \approx \frac{\eta}{\Delta t} AB\vartheta(r) \frac{\eta\vartheta_1(r)}{\Delta t} A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{33}$$

where

$$\vartheta(r) = \begin{bmatrix} \vartheta_2(r) \\ \vartheta_3(r) \\ \vdots \\ \vartheta_{n_t}(r) \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & -1 & 1 \end{bmatrix}$$

For space discretization, we use the pseudo-spectral collocation method. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ are pseudo-spectral-collocation operators for the approximation of first-order and second-order derivatives, respectively. We make the partition of $r \in [0,1]$ as $\{r_1, r_2, \cdots, r_{n_r}\}$. The sizes of D_1 and D_2 are n_r. To deal with a second-order boundary value problem, we construct the following differential operator for our problem.

$$\begin{matrix} \mathbf{D} = \mathbf{D}_2(2:n_r-1, 2:n_r-1) + diag(1./\mathbf{r}(2:n_r-1)) \\ \mathbf{D}_1(2:n_r-1, 2:n_r-1) - diag(p^2./\mathbf{r}(2:n_r-1).^2) \end{matrix}, \tag{34}$$

where ./, are the element-wise operation of division and power in Matlab format. The meaning of $2:n_r-1$ is $2,3,\cdots,n_r-1$. To accommodate the boundary conditions, we define two column vectors

$$\begin{matrix} \mathbf{d}_1 = \mathbf{D}_2(:,1) + diag(1./\mathbf{r}(2:n_r-1))\mathbf{D}_1(:,1) \\ \mathbf{d}_2 = \mathbf{D}_2(:,n_r) + diag(1./\mathbf{r}(2:n_r-1))\mathbf{D}_1(:,n_r) \end{matrix}, \tag{35}$$

where $\mathbf{D}_{1or2}(:,i)$ means ith column of $\mathbf{D}_{1or2}(:,i)$ matrix. We can write down the discretization of our problem in this form

$$\mathbf{M} = \mathbf{I} - \frac{\Delta t}{\eta}(\mathbf{D} \otimes \mathbf{C}),$$

$$\mathbf{m} = \frac{\Delta t}{\eta}(\mathbf{d}_1 \otimes \mathbf{C}\vartheta_1 + \mathbf{d}_2 \otimes \mathbf{C}\vartheta_{n_r} + \phi \otimes \mathbf{B}^{-1}\mathbf{e}_1), \tag{36}$$

where $\mathbf{C} = (\mathbf{AB})^{-1}$, $\vartheta = [\vartheta_2, \vartheta_3, \cdots, \vartheta_{n_r-1}]^T$, $\theta_i = [\vartheta(r_i,t_2), \vartheta(r_i,t_3), \cdot, \vartheta(r_i,t_{n_t})]^T$, $\mathbf{e}_1$ is the first column identity matrix $\mathbf{I}$, ϑ_1 and ϑ_{n_r} are vectors for boundary conditions, and ϕ is a vector that implements initial condition. The nonlinear equation is discretized by using a pseudo-spectral collocation method [26] based on Newton Raphson Method [27–30]. It is well-known that the pseudo-spectral collocation method offers high accuracy in the approximation of derivatives. To perform numerical simulation, we take the size of the system of nonlinear equation $n = 100$.The total number of equations is 200; we are solving a system of nonlinear coupled equations. In all the simulations, we take $\mathbf{0}$ as an initial guess, and on average, we get numerical accuracy in the solution of a system of nonlinear equations.

Finally, by solving a system of linear equations

$\mathbf{M}\vartheta = \mathbf{m}$,

we get the solution vector

ϑ.

4. Comparison between the analytical and numerical solution

For the validity of our analytical solution, which is obtained by the Laplace, transformation is compared with the numerical solution obtained with the help of the pseudo-spectral collocation method. The detail of this method is already mentioned in the previous section. We have constructed Fig. 1 to highlight the comparison of both solutions and noted that both solutions are very closed to each other.

5. Numerical results and discussion

This section deals with the interpretation of the analytical results for the solute concentration with the graphical illustrations. We have used Mathcad software for numerical results. Fig. 2 illustrates the effects of the fractional parameter on the solute concentration $\psi_c(r, t)$. It has been seen that neat the surface of the cylinder there exists values of the fractional parameter α for which the solute concentration with respect to the fractional advection–diffusion process is higher as compared with the ordinary model. Moreover, the converse behavior has been also observed for some other values of the fractional parameter α. In conclusion, the fractional parameter α which indicates the memory effects (i.e. the effects on the history of the flow) has a significant influence on the solute concentration as compared with the ordinary model of the advection–diffusion process. Fig. 3 describes the effects of Pe(Peclet number) on the concentration $\psi_c(r, t)$. It has been observed that drift velocity changes equivalent to the change in the Peclet number. The domination of the (diffusion or the advection) transport process is shown by the Peclet number. A decrease in solute concentration has been observed with the increase in Peclet number whereas the concentration increases with the radial coordinate r. For $Pe = 1$ (p = 0.50), the solute concentration $\psi_c(r, t)$increases almost linear.

6. Conclusions

In the present research, we use the first-order approximation of ordinary derivatives by using the forward difference method. It is of interest in future research to use a higher-order approximation for ordinary derivatives. As the derivatives in space are ordinary derivatives, we use highly accurate pseudo-spectral collocation approximation for them.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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