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FIXED POINT RESULTS FOR $F - \mu_s - \rho_s^*$ CONTRACTION IN QUASI b-METRIC SPACES WITH SOME APPLICATIONS

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ABSTRACT. In this article, we discuss a recent generalization of a quasi metric space and introduce $F - \mu_s - \rho_s^*$ contraction which is a generalization of many recently announced contractions. Fixed point results for some of such contractions have been obtained. An example of our main result is also given which shows that how our result can be used when others fail. We achieve results endowed with a graph. Results in ordered left K-sequentially complete quasi b-metric space have been established. We consider applications of our main results for the existence of a unique common solution for a system of integral equations and of a unique solution for functional equations that arises in dynamic programming.

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Keywords and Phrases: Common fixed point; multivalued mapping; $\{TS(x_n)\}$ sequence; $F - \mu_s - \rho_s^*-$contraction; complete quasi b-metric space; open ball; graph; partial order; integral equations; dynamic programming.

1. Introduction and preliminaries

One of the generalizations of the metric space is the quasi metric space that was introduced by Wilson [48]. The commutativity condition does not hold in general in a quasi metric space, see [9, 14, 16, 20, 35, 50, 51]. Several authors extended and generalized this concept in different ways, see [4, 5, 14, 25, 29, 31, 32, 63]. The quasi b-metric space, see [26, 44] is a generalization of a quasi metric space as well as a b-metric space, see [3, 43, 62]. In this paper, we are using quasi b-metric spaces

Nadler [36] presented fixed point theorem for multivalued mappings and generalized the results for single-valued mappings. Since then, an interesting and rich fixed point theory for such mappings was developed in many directions, see [13, 15, 46, 47]. Fixed point results of multivalued mappings have applications in engineering, control

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theory, differential equations, games and economics, see [12, 21]. In this paper, we are using multivalued mappings.

Wardowski [49] introduced a mapping F and a contraction to obtain a fixed point result. For more results on this direction, see [1, 10, 33, 34, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61]. Recently, Rasham et al. [41] proved some fixedp oint results by using only one condition out of three conditions of Wardowski. In this paper, we are using the methodology given in [41].

Arshad et al. [8] observed that there were mappings which had fixed points but there were no results to ensure the existence of a fixed point of such mappings. They introduced a condition on closed ball to achieve common fixed points for such mappings. For further results on closed ball, see [40, 41, 45]. In this paper, we are using open ball instead of closed ball.

Ran and Reurings [39] and Nieto et al. [37] gave an extension to the results in fixed point theory and obtained results in partially ordered sets. Altun et al. [6] introduced a new approach to common fixed point of mappings, satisfying a generalized contraction with a new restriction of order, in a complete ordered metric spaces. For more results in ordered spaces see [22, 23, 24]. Asl et al. [11] gave the idea of α-ψ contractive multifunctions (see also [2, 27, 42]) and generalized the restriction of order. In this paper, we have modified the result of Altun et al. [6] by introducing $F-\mu_s-\rho_s^*$ contractions and generalizing their restriction of order.

First, we recall the following definitions and results which will be useful to understand the paper. Berinde [17] introduced the class of b-comparison functions, see also [43, 45].

Definition 1.1. [43] Let $s \geq 1$ and $\mu_s : [0,\infty) \to [0,\infty)$ be a function, which satisfies:

(Ψ_{s1}) μ_s is non-decreasing.

(Ψ_{s2}) For all $t > 0$, we have $\sum\limits_{k=0}^{\infty} s^k \mu_s^k(t) < \infty$, where μ_s^k is the k^{th} iterate of μ_s.

Then the function μ_s is called b-comparison function. Let $s \geq 1$, then $\mu_s(t) = bt$, $t \in \mathbb{R}^+$ with $0 < b < \frac{1}{s}$ is a b-comparison function. For each value of "s" in the given example, we can obtain infinitely many b-comparison functions by taking different values of "b". The set of all b-comparison functions is denoted by Ψ_s. If we take $s = 1$, then μ_s is called (c)-comparison function. If $\mu(t) = \frac{t}{1+t}$, then μ is a (c)-comparison function. The set of all (c)-comparison functions is denoted by Ψ.

Lemma 1.2. [43] *Let* $\mu_s \in \Psi_s$. *Then*

(i) $s\mu_s(t) < t$, *for all* $t > 0$,

(ii) $\mu_s(0) = 0$.

Clearly $s\mu_s(t) < t$ *for all* $t > 0$ *implies* $s^{n+1}\mu_s^{n+1}(t) < s^n\mu_s^n(t)$.

Definition 1.3. [44] Let X be a nonempty set, $s \geq 1$, $a, y, z \in X$ and $q_s : X \times X \to [0, \infty)$ be a function, which satisfies:

(q_1) $q_s(a, y) = 0$ if and only if $a = y$,

(q_2) $q_s(a, y) \leq s\left[q_s(a, z) + q_s(z, y)\right].$

Then q_s is called a quasi b-metric and the pair (X, q_s) is called a quasi b-metric space. The number s is called the coefficient of (X, q_s). For $a \in X$ and $\varepsilon > 0$, $B_{q_s}(a, \varepsilon) = \{y \in X : q_s(a, y) < \varepsilon \text{ and } q_s(y, a) < \varepsilon\}$ and $\overline{B_{q_s}(a, \varepsilon)} = \{y \in X : q_s(a, y) \leq \varepsilon \text{ and } q_s(y, a) \leq \varepsilon\}$ are open ball and closed ball in (X, q_s) respectively.

Example 1.4. [44] Let $X = \{1, 2, 3\}$. Define the function q_s on $X \times X$ as $q_s(n, m) = 1/n^2$ for all $n > m$, $q(n, m) = 1$ for $n < m$, and $q(n, m) = 0$, for $n = m$, with $(n, m) \neq (1, 2)$ and $q_s(1, 2) = 16/9$. Then (X, q_s) is a quasi b-metric space with coefficient $s = 2$. It is neither a b-metric space since $q_s(1, 2) = 16/9 \neq q_s(2, 1) = 1/4$, nor a quasi metric space since $q_s(1, 2) = 16/9 > 10/9 = q_s(1, 3) + q(3, 2)$.
Beg et al. [16] introduced the notion of left (right) K -Cauchy sequence and left (right) K-sequentially complete spaces.

Definition 1.5. [16] Let (X, q_s) be a quasi b-metric space.

(a) A sequence $\{a_n\}$ in (X, q_s) is called left (right) K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $q_s(a_m, a_n) < \varepsilon$ (respectively $q_s(a_n, a_m) < \varepsilon$) for all $m > n \geq n_0$.
(b) A sequence $\{a_n\}$ in (X, q_s) converges to a, if $\lim_{n\to\infty} q_s(a_n, a) = \lim_{n\to\infty} q_s(a, a_n) = 0$. In this case, the point a is called a limit of the sequence $\{a_n\}$.
(c) (X, q_s) is called left (right) K -sequentially complete if every left (right) K -Cauchy sequence in (X, q_s) converges to a point $a \in X$.

Definition 1.6. [47] Let (X, q_s) be a quasi b-metric space. Let K be a non empty subset of X and let $a \in X$. An element $y_0 \in K$ is called a best approximation in K, if

$$q_s(a, K) = q_s(a, y_0), \text{ where } q_s(a, K) = \inf\{q_s(a, y),\ y \in K\},$$

$$\text{and } q_s(K, a) = q_s(y_0, a), \text{ where } q_s(K, a) = \inf\{q_s(y, a),\ y \in K\}.$$

If each $a \in X$ has at least one best approximation in K, then K is called a proximinal set. We denote by $P(X)$, the set of all proximinal subsets of X.

Definition 1.7. [47] The function $H_{q_s} : P(X) \times P(X) \to [0, \infty)$, defined by

$$H_{q_s}(A, B) = \max\left\{\sup_{x\in A} q_s(x, B), \sup_{y\in B} q_s(A, y)\right\},$$

is called quasi Hausdorff b-metric on $P(X)$. Also $(P(X), H_{q_s})$ is known as quasi Hausdorff b-metric space.

Lemma 1.8. [47] *Let (X, q_s) be a quasi b-metric space. Let $(P(X), H_{q_s})$ be a quasi Hausdorff b-metric space on $P(X)$. Then, for all $A, B \in P(X)$ and for each $a \in A$, there exists $b_a \in B$ such that $H_{q_s}(A, B) \geq q_s(a, b_a)$ and $H_{q_s}(B, A) \geq q_s(b_a, a)$, where $q_s(a, B) = q_s(a, b_a)$ and $q_s(B, a) = q_s(b_a, a)$*

Definition 1.9. Let X be a non empty set, $s \geq 1$ and $\rho_s : X \times X \to [0, +\infty)$ be a mapping. Let $M \subseteq X$, define $\rho_s^*(x, M) = \inf \{\rho_s(x, a), a \in M\}$ and $\rho_s^*(M, y) = \inf \{\rho_s(b, y), b \in M\}$.

Definition 1.10. Let Ω be the family of all strictly increasing functions $F : \mathbb{R}_+ \to \mathbb{R}$, that is for all $x, y \in \mathbb{R}_+$, if $x < y$, then $F(x) < F(y)$.

2. Main results

Let (X, q_s) be a quasi b-metric space, $a_0 \in X$ and $S, T : X \to P(X)$ be the multivalued mappings on X. Let $a_1 \in Sa_0$ such that $q_s(a_0, Sa_0) = q_s(a_0, a_1)$ and $q_s(Sa_0, a_0) = q_s(a_1, a_0)$. Now, for $a_1 \in X$, there exists $a_2 \in Ta_1$ such that $q_s(a_1, Ta_1) = q_s(a_1, a_2)$ and $q_s(Ta_1, a_1) = q_s(a_2, a_1)$. Continuing this process, we construct a sequence $\{a_n\}$ of points in X such that $a_{2n+1} \in Sa_{2n}$, and $a_{2n+2} \in Ta_{2n+1}$ with $q_s(a_{2n}, Sa_{2n}) = q_s(a_{2n}, a_{2n+1})$, $q_s(Sa_{2n}, a_{2n}) = q_s(a_{2n+1}, a_{2n})$ and $q_s(a_{2n+1}, Ta_{2n+1}) = q_s(a_{2n+1}, a_{2n+2})$, $q_s(Ta_{2n+1}, a_{2n+1}) = q_s(a_{2n+2}, a_{2n+1})$. We denote this iterative sequence by $\{TS(a_n)\}$ and say that $\{TS(a_n)\}$ is a sequence in X generated by a_0.

Definition 2.1. Let (X, q_s, s) be a left K-sequentially complete quasi b-metric space, $\rho_s : X \times X \to [0, +\infty)$ and $S, T : X \to P(X)$ be two multivalued mappings. The pair (S, T) is called $F - \mu_s - \rho_s^*$ contraction on the intersection of an open ball and a sequence if $\mu_s \in \Psi$, $F \in \Omega$, $a_0 \in X$, $r, \tau > 0$, $a, y \in B_{q_s}(a_0, r) \cap \{TS(a_n)\}$, $\rho_s^*(Sy, y) \geq s$, $\rho_s^*(a, Sa) \geq s$, $q_s(a, Ty) + q_s(y, Sa) \neq 0$ and $\max\{H_{q_s}(Sa, Ty), H_{q_s}(Ty, Sa), Q_s(a, y), Q_s(y, a)\} > 0$, then

$$\tau + \max\{F(H_{q_s}(Sa, Ty)), F(H_{q_s}(Ty, Sa))\} \leq F(\mu_s(Q_s(a, y))), \tag{2.1}$$

where

$$Q_s(a, y) = \max\left\{q_s(a, y), q_s(a, Sa), \frac{q_s(a, Sa)\, q_s(a, Ty) + q_s(y, Ty)\, q_s(y, Sa)}{q_s(a, Ty) + q_s(y, Sa)}\right\}.$$

Also, if $q_s(a, Ty) + q_s(y, Sa) = 0$, then

$$\max\{H_{q_s}(Sa, Ty), H_{q_s}(Ty, Sa), Q_s(a, y), Q_s(y, a)\} = 0.$$

Moreover,

$$\sum_{i=0}^{j} s^{i+1}\left[\max\left\{\mu_s^i(q_s(a_1, a_0)), \mu_s^i(q_s(a_0, a_1))\right\}\right] < r, \text{ for all } j \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Theorem 2.2. *Let (X, q_s, s) be a left K-sequentially complete quasi b-metric space, $\rho_s : X \times X \to [0, +\infty)$, $S, T : X \to P(X)$ and (S, T) be $F - \mu_s - \rho_s^*$ contraction on open ball. Suppose that the following assumptions hold:*

(i) *If $a \in B_{q_s}(a_0, r)$,*

$$\begin{aligned} \textbf{(a)}\ \rho_s^*(a, Sa) &\geq s,\ q_s(a, Sa) = q_s(a, y) \text{ and} \\ q_s(Sa, a) &= q_s(y, a) \text{ implies } \rho_s^*(Sy, y) \geq s, \end{aligned}$$

$$\begin{aligned} \textbf{(b)}\ \rho_s^*(Sa, a) &\geq s,\ q_s(a, Ta) = q_s(a, y) \text{ and} \\ q_s(Ta, a) &= q_s(y, a) \text{ implies } \rho_s^*(y, Sy) \geq s. \end{aligned}$$

(ii) *The set $G(S) = \{a : \rho_s^*(a, Sa) \geq s$ and $a \in B_{q_s}(a_0, r)\}$ is closed and contains a_0.*

Then the subsequence $\{a_{2n}\}$ of $\{TS(a_n)\}$ is a sequence in $G(S)$ and $\{a_{2n}\} \to a^* \in G(S)$ and $q_s(a^*, a^*) = 0$. Also, if inequality (2.1) holds for a^*. Then T and S have a common fixed point a^* in $B_{q_s}(a_0, r)$.

Proof. Consider the sequence $\{TS(a_n)\}$ generated by a_0. By assumption (ii), $G(S)$ contains a_0, therefore $\rho_s^*(a_0, Sa_0) \geq s$ and $a_0 \in B_{q_s}(a_0, r)$. Then there exists $a_1 \in Sa_0$ such that $q_s(a_0, Sa_0) = q_s(a_0, a_1)$ and $q_s(Sa_0, a_0) = q_s(a_1, a_0)$. From condition (i) $\rho_s^*(Sa_1, a_1) \geq s$. By (2.2), we have

$$\max\{q_s(a_1, a_0), q_s(a_0, a_1)\} \leq \sum_{i=0}^{j} s^{i+1}\left[\max\left\{\mu_s^i(q_s(a_1, a_0)), \mu_s^i(q_s(a_0, a_1))\right\}\right] < r.$$

That is $q_s(a_1, a_0) < r$, and $q_s(a_0, a_1) < r$. Hence $a_1 \in B_{q_s}(a_0, r)$. Also

$$q_s(a_1, Ta_1) = q_s(a_1, a_2) \text{ and } q_s(Ta_1, a_1) = q_s(a_2, a_1).$$

As $\rho_s^*(Sa_1, a_1) \geq s$, so from assumption (i), we have $\rho_s^*(a_2, Sa_2) \geq s$. Now, by Lemma 1.8, we have

$$q_s(a_{2i}, a_{2i+1}) \leq H_{q_s}(Ta_{2i-1}, Sa_{2i}),\ q_s(a_{2i+1}, a_{2i}) \leq H_{q_s}(Sa_{2i}, Ta_{2i-1}) \tag{2.3}$$

and

$$q_s(a_{2i+1}, a_{2i+2}) \leq H_{q_s}(Sa_{2i}, Ta_{2i+1}),\ q_s(a_{2i+2}, a_{2i+1}) \leq H_{q_s}(Ta_{2i+1}, Sa_{2i}). \tag{2.4}$$

By the triangle inequality, we have

$$q_s(a_0, a_2) \leq s[q_s(a_0, a_1) + q_s(a_1, a_2)]. \tag{2.5}$$

By using (2.4), we have

$$\tau + F(q_s(a_1, a_2)) \leq \tau + F(H_{q_s}(Sa_0, Ta_1)),$$

$$\tau + F(q_s(a_1, a_2)) \leq \tau + \max\{F(H_{q_s}(Sa_0, Ta_1)), F(H_{q_s}(Ta_1, Sa_0))\}. \tag{2.6}$$

Now, let $a_{2i'}, a_{2i'+1}$ be two consecutive elements of the sequence $\{TS(a_n)\}$. Clearly, if

$$\max\{H_{q_s}(Sa_{2i'}, Ta_{2i'+1}), H_{q_s}(Ta_{2i'+1}, Sa_{2i'}), Q_s(a_{2i'}, a_{2i'+1}), Q_s(a_{2i'+1}, a_{2i'}) \not> 0,$$

for some $i' \in \mathbb{N} \cup \{0\}$, or if $q_s(a_{2i'}, Ta_{2i'+1}) + q_s(a_{2i'+1}, Sa_{2i'}) = 0$, then

$$H_{q_s}(Sa_{2i'}, Ta_{2i'+1}) = H_{q_s}(Ta_{2i'+1}, Sa_{2i'}) = Q_s(a_{2i'}, a_{2i'+1}) = Q_s(a_{2i'+1}, a_{2i'}) = 0.$$

If $Q_s(a_{2i'}, a_{2i'+1}) = 0$, then $q_s(a_{2i'}, a_{2i'+1}) = 0$. Also, if $Q_s(a_{2i'+1}, a_{2i'}) = 0$, then $q_s(a_{2i'+1}, a_{2i'}) = 0$ so, $a_{2i'+1} = a_{2i'}$ and $a_{2i'} \in Sa_{2i'}$. Now, $H_{q_s}(Sa_{2i'}, Ta_{2i'+1}) = 0$ implies $q_s(a_{2i'+1}, Ta_{2i'+1}) = 0$ and $H_{q_s}(Ta_{2i'+1}, Sa_{2i'}) = 0$ implies $q_s(Ta_{2i'+1}, a_{2i'+1}) = 0$. So, $a_{2i'+1} \in Ta_{2i'+1}$ and hence $a_{2i'}$ is a common fixed point of S and T. So, the proof is done. Now, suppose

$$\max\{H_{q_s}(Sa_{2i}, Ta_{2i+1}), H_{q_s}(Ta_{2i+1}, Sa_{2i}), Q_s(a_{2i}, a_{2i+1}), Q_s(a_{2i+1}, a_{2i})\} > 0,$$

and $q_s(a_{2i}, Ta_{2i+1}) + q_s(a_{2i+1}, Sa_{2i}) \neq 0$ for all $i \in \{0\} \cup \mathbb{N}$. As $a_0, a_1 \in B_{q_s}(a_0, r) \cap \{TS(a_n)\}$, $\rho_s^*(Sa_1, a_1) \geq s$, and $\rho_s^*(a_0, Sa_0) \geq s$, by using (2.1) in (2.6), we have

$$\begin{aligned}
\tau + F(q_s(a_1, a_2)) &\leq F(\mu_s(Q_s(a_0, a_1))) \\
&= F\Bigg(\mu_s\Bigg(\max\Bigg\{q_s(a_0, a_1), q_s(a_0, a_1), \\
&\qquad \frac{q_s(a_0, a_1)\, q_s(a_0, Ta_1) + q_s(a_1, a_2)(0)}{q_s(a_0, Ta_1) + (0)}\Bigg\}\Bigg)\Bigg) \\
&= F(\mu_s(q_s(a_0, a_1))).
\end{aligned}$$

Since F is strictly increasing and $\tau > 0$, $q_s(a_1, a_2) < \mu_s(q_s(a_0, a_1))$. Now, inequality (2.5)

$$\begin{aligned}
q_s(a_0, a_2) &< s[q_s(a_0, a_1) + \mu_s(q_s(a_0, a_1))] \\
&\leq s[\max\{q_s(a_1, a_0), q_s(a_0, a_1)\}] \\
&\quad + s[\max\{\mu_s(q_s(a_1, a_0)), \mu_s(q_s(a_0, a_1))\}]
\end{aligned}$$

$$\leq \sum_{i=0}^{1} s^{i+1}\left[\max\left\{\mu_s^i(q_s(a_1, a_0)), \mu_s^i(q_s(a_0, a_1))\right\}\right] < r.$$

Now, by using (2.4), we have

$$\begin{aligned}
\tau + F(q_s(a_2, a_1)) &\leq \tau + F(H_{q_s}(Ta_1, Sa_0)) \\
&\leq \tau + \max\{F(H_{q_s}(Ta_1, Sa_0)), F(H_{q_s}(Sa_0, Ta_1))\}.
\end{aligned}$$

As $a_1, a_0 \in B_{q_s}(a_0, r) \cap \{TS(a_n)\}$, $\rho_s^*(a_0, Sa_0) \geq s$ and $\rho_s^*(Sa_1, a_1) \geq s$, by (2.1), we have

$$\tau + F(q_s(a_2, a_1)) \leq F(\mu_s(Q_s(a_0, a_1))) \leq F(\mu_s(q_s(a_0, a_1))).$$

Since F is strictly increasing and $\tau > 0$,

$$q_s(a_2, a_1) < \mu_s(\max\{q_s(a_1, a_0), q_s(a_0, a_1)\}).$$

Now, by the triangle inequality

$$d\left(a_2,a_0\right) \leq \sum_{i=0}^{j} s^{i+1}\left[\max\left\{\mu_s^i\left(q_s\left(a_1,a_0\right)\right),\mu_s^i\left(q_s\left(a_0,a_1\right)\right)\right\}\right] < r.$$

It follows that, $q_s\left(a_0,a_2\right) < r$ and $q_s\left(a_2,a_0\right) < r$. So $a_2 \in B_{q_s}(a_0,r)$. Also

$$q_s\left(a_2,Sa_2\right) = q_s\left(a_2,a_3\right) \text{ and } q_s\left(Sa_2,a_2\right) = q_s\left(a_3,a_2\right).$$

As $\rho_s^*\left(a_2,Sa_2\right) \geq s$, so from assumption (i), we have $\rho_s^*\left(Sa_3,a_3\right) \geq s$. Let $a_3,...a_j \in B_{q_s}(a_0,r)$, and $\rho_s^*\left(a_0,Sa_0\right) \geq s$, $\rho_s^*\left(Sa_1,a_1\right) \geq s$, $\rho_s^*\left(a_2,Sa_2\right) \geq s$, $\rho_s^*\left(Sa_3,a_3\right) \geq s,\cdots$, $\rho_s^*\left(Sa_{j+1},a_{j+1}\right) \geq s$, for some $j \in \mathbb{N}$, where $j = 2i$, $i = 1,2,3,...,\frac{j}{2}$. Now by using (2.3), we have

$$\begin{aligned}\tau + F\left(q_s\left(a_{2i},a_{2i+1}\right)\right) &\leq \tau + F\left(H_{q_s}\left(Ta_{2i-1},Sa_{2i}\right)\right)\\ &\leq \tau + \max\left\{F\left(H_{q_s}\left(Ta_{2i-1},Sa_{2i}\right)\right),F\left(H_{q_s}\left(Sa_{2i},Ta_{2i-1}\right)\right)\right\}.\end{aligned}$$

As $a_{2i-1},a_{2i} \in B_{q_s}(a_0,r) \cap \{TS\left(a_n\right)\}$, $\rho_s^*\left(a_{2i},Sa_{2i}\right) \geq s$, $\rho_s^*\left(Sa_{2i-1},a_{2i-1}\right) \geq s$ and $\max\{H_{q_s}(Ta_{2i-1},Sa_{2i}),H_{q_s}(Sa_{2i},Ta_{2i-1}),Q_s(a_{2i-1},a_{2i}),Q_s(a_{2i},a_{2i-1})\} > 0$, then by (2.1), we have

$$\begin{aligned}\tau + F\left(q_s\left(a_{2i},a_{2i+1}\right)\right) &\leq F(\mu_s(Q_s(a_{2i},a_{2i-1})))\\ &\leq F\left(\mu_s\left(\max\left\{q_s\left(a_{2i},a_{2i-1}\right),q_s\left(a_{2i},a_{2i+1}\right),\right.\right.\right.\\ &\qquad \left.\left.\left.\frac{q_s\left(a_{2i},a_{2i+1}\right)(0) + q_s\left(a_{2i-1},a_{2i}\right)q_s\left(a_{2i-1},Sa_{2i}\right)}{0 + q_s\left(a_{2i-1},Sa_{2i}\right)}\right\}\right)\right)\\ &= F\left(\mu_s\left(\max\{q_s\left(a_{2i},a_{2i-1}\right),q_s\left(a_{2i},a_{2i+1}\right),q_s\left(a_{2i-1},a_{2i}\right)\}\right)\right).\end{aligned}$$

If $\max\{q_s\left(a_{2i},a_{2i-1}\right),q_s\left(a_{2i},a_{2i+1}\right),q_s\left(a_{2i-1},a_{2i}\right)\} = q_s\left(a_{2i},a_{2i+1}\right)$, then

$$\tau + F\left(q_s\left(a_{2i},a_{2i+1}\right)\right) \leq F\left(\mu_s\left(q_s\left(a_{2i},a_{2i+1}\right)\right)\right),$$

which implies $q_s\left(a_{2i},a_{2i+1}\right) < \mu_s\left(q_s\left(a_{2i},a_{2i+1}\right)\right) < s\mu_s\left(q_s\left(a_{2i},a_{2i+1}\right)\right)$. This is contradiction to the fact $s\mu_s(t) < t$, so $\max\{q_s\left(a_{2i},a_{2i-1}\right),q_s\left(a_{2i},a_{2i+1}\right),q_s\left(a_{2i-1},a_{2i}\right)\} \neq q_s\left(a_{2i},a_{2i+1}\right)$. Therefore, we have

$$\tau + F\left(q_s\left(a_{2i},a_{2i+1}\right)\right) \leq F(\mu_s\left(\max\left\{q_s\left(a_{2i-1},a_{2i}\right),q_s\left(a_{2i},a_{2i-1}\right)\right\}\right)).$$

Since F is strictly increasing and $\tau > 0$,

$$q_s\left(a_{2i},a_{2i+1}\right) < \max\left\{\mu_s\left(q_s\left(a_{2i-1},a_{2i}\right)\right),\mu_s\left(q_s\left(a_{2i},a_{2i-1}\right)\right)\right\}. \tag{2.7}$$

Now, by (2.4), we have

$$\tau + F\left(q_s\left(a_{2i-1},a_{2i}\right)\right) \leq \tau + F\left(H_{q_s}\left(Sa_{2i-2},Ta_{2i-1}\right)\right)$$

$$\leq \tau + F\left(\max\left\{H_{q_s}\left(Sa_{2i-2},Ta_{2i-1}\right),H_{q_s}\left(Ta_{2i-1},Sa_{2i-2}\right)\right\}\right).$$

As $a_{2i-1}, a_{2i-2} \in B_{q_s}(a_0, r) \cap \{TSa_n\}$, $\rho_s^*(Sa_{2i-1}, a_{2i-1}) \geq s$, $\rho_s^*(a_{2i-2}, Sa_{2i-2}) \geq s$ and $\max\{H_{q_s}(Sa_{2i-2}, Ta_{2i-1}), H_{q_s}(Ta_{2i-1}, Sa_{2i-2}), Q_s(a_{2i-2}, a_{2i-1}), Q_s(a_{2i-1}, a_{2i-2}) > 0$, by (2.1), we have

$$\begin{aligned} & \tau + F\left(q_s\left(a_{2i-1}, a_{2i}\right)\right) \\ \leq\ & F(\mu_s\left(Q_s(a_{2i-2}, a_{2i-1})\right)) \\ =\ & F\left(\mu_s\left(\max\left\{q_s\left(a_{2i-2}, a_{2i-1}\right), q_s\left(a_{2i-2}, a_{2i-1}\right), q_s\left(a_{2i-2}, a_{2i-1}\right)\right\}\right)\right) \\ =\ & F\left(\mu_s\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right)\right). \end{aligned}$$

Since F is strictly increasing and $\tau > 0$,

$$q_s\left(a_{2i-1}, a_{2i}\right) < \mu_s\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right).$$

$$q_s\left(a_{2i-1}, a_{2i}\right) < \mu_s\left(\max\left\{q_s\left(a_{2i-1}, a_{2i-2}\right), q_s\left(a_{2i-2}, a_{2i-1}\right)\right\}\right).$$

As μ_s is non decreasing function, so

$$\mu_s\left(q_s\left(a_{2i-1}, a_{2i}\right)\right) < \max\left\{\mu_s^2\left(q_s\left(a_{2i-1}, a_{2i-2}\right)\right), \mu_s^2\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right)\right\}. \tag{2.8}$$

Now, by (2.4), we have

$$\tau + F\left(q_s\left(a_{2i}, a_{2i-1}\right)\right) \leq \tau + F\left(H_{q_s}\left(Ta_{2i-1}, Sa_{2i-2}\right)\right)$$

$$\leq \tau + F\left(\max\left\{H_{q_s}\left(Sa_{2i-2}, Ta_{2i-1}\right), H_{q_s}\left(Ta_{2i-1}, Sa_{2i-2}\right)\right\}\right).$$

By (2.1), we have

$$\begin{aligned} \tau + F\left(q_s\left(a_{2i}, a_{2i-1}\right)\right) &\leq F(\mu_s\left(Q_s(a_{2i-2}, a_{2i-1})\right)) \\ &= F\left(\mu_s\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right)\right). \end{aligned}$$

Since F is strictly increasing and $\tau > 0$,

$$q_s\left(a_{2i}, a_{2i-1}\right) < \mu_s\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right),$$

$$q_s\left(a_{2i}, a_{2i-1}\right) < \mu_s\left(\max\left\{q_s\left(a_{2i-1}, a_{2i-2}\right), q_s\left(a_{2i-2}, a_{2i-1}\right)\right\}\right).$$

As μ_s is non decreasing function, so

$$\mu_s\left(q_s\left(a_{2i}, a_{2i-1}\right)\right) < \max\left\{\mu_s^2\left(q_s\left(a_{2i-1}, a_{2i-2}\right)\right), \mu_s^2\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right)\right\}. \tag{2.9}$$

Now, By (2.8) and (2.9), we have

$$\max\left\{\mu_s\left(q_s\left(a_{2i-1}, a_{2i}\right)\right), \mu_s\left(q_s\left(a_{2i}, a_{2i-1}\right)\right)\right\}$$

$$< \max\left\{\mu_s^2\left(q_s\left(a_{2i-1}, a_{2i-2}\right)\right), \mu_s^2\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right)\right\}. \tag{2.10}$$

By (2.10) and (2.7), we have

$$q_s\left(a_{2i}, a_{2i+1}\right) < \max\left\{\mu_s^2\left(q_s\left(a_{2i-1}, a_{2i-2}\right)\right), \mu_s^2\left(q_s\left(a_{2i-2}, a_{2i-1}\right)\right)\right\}. \tag{2.11}$$

Now, by using (2.3), we have

$$\tau + F\left(q_s(a_{2i-2}, a_{2i-1})\right) \leq \tau + F\left(H_{q_s}(Ta_{2i-3}, Sa_{2i-2})\right)$$

$$\leq \tau + \max\left\{F\left(H_{q_s}(Ta_{2i-3}, Sa_{2i-2})\right), F\left(H_{q_s}(Sa_{2i-2}, Ta_{2i-3})\right)\right\}.$$

As $a_{2i-3}, a_{2i-2} \in B_{q_s}(a_0, r) \cap \{TSa_n\}$, $\rho_s^*(Sa_{2i-3}, a_{2i-3}) \geq s$, $\rho_s^*(a_{2i-2}, Sa_{2i-2}) \geq s$ and $\max\{H_{q_s}(Sa_{2i-2}, Ta_{2i-1}), H_{q_s}(Ta_{2i-1}, Sa_{2i-2}), Q_s(a_{2i-2}, a_{2i-1}), Q_s(a_{2i-1}, a_{2i-2}) > 0$, by (2.1), we have

$$\begin{aligned} & \tau + F\left(q_s(a_{2i-2}, a_{2i-1})\right) \\ \leq\ & F(\mu_s\left(Q_s(a_{2i-2}, a_{2i-3})\right)) \\ =\ & F\left(\mu_s\left(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s\left(a_{2i-2}, a_{2i-1}\right), q_s\left(a_{2i-3}, a_{2i-2}\right)\right\}\right)\right) \\ =\ & F\left(\mu_s\left(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s\left(a_{2i-3}, a_{2i-2}\right)\right\}\right)\right), \end{aligned}$$

which implies that

$$q_s(a_{2i-2}, a_{2i-1}) < \mu_s(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s(a_{2i-3}, a_{2i-2})\right\}),$$

$$\mu_s^2 q_s(a_{2i-2}, a_{2i-1}) < \mu_s^3(\max\left\{q_s(a_{2i-3}, a_{2i-2}), q_s(a_{2i-2}, a_{2i-3})\right\}). \tag{2.12}$$

Now, by using (2.3), we have

$$\tau + F\left(q_s(a_{2i-1}, a_{2i-2})\right) \leq \tau + F\left(H_{q_s}(Sa_{2i-2}, Ta_{2i-3})\right)$$

$$\leq \tau + \max\left\{F\left(H_{q_s}(Ta_{2i-3}, Sa_{2i-2})\right), F\left(H_{q_s}(Sa_{2i-2}, Ta_{2i-3})\right)\right\}.$$

As $a_{2i-3}, a_{2i-2} \in B_{q_s}(a_0, r) \cap \{TSa_n\}$, $\rho_s^*(Sa_{2i-3}, a_{2i-3}) \geq s$, $\rho_s^*(a_{2i-2}, Sa_{2i-2}) \geq s$ and $\max\{H_{q_s}(Sa_{2i-2}, Ta_{2i-1}), H_{q_s}(Ta_{2i-1}, Sa_{2i-2}), Q_s(a_{2i-2}, a_{2i-1}), Q_s(a_{2i-1}, a_{2i-2}) > 0$, by (2.1), we have

$$\tau + F\left(q_s(a_{2i-1}, a_{2i-2})\right) \leq F(\mu_s\left(Q_s(a_{2i-2}, a_{2i-3})\right))$$

$$= F\left(\mu_s\left(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s\left(a_{2i-2}, a_{2i-1}\right), q_s\left(a_{2i-3}, a_{2i-2}\right)\right\}\right)\right).$$

Now, by using (2.12), we have

$$\begin{aligned} q_s(a_{2i-2}, a_{2i-1}) &< \mu_s(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s(a_{2i-3}, a_{2i-2})\right\}) \\ &\leq s\mu_s(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s(a_{2i-3}, a_{2i-2})\right\}) \\ &< \max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s(a_{2i-3}, a_{2i-2})\right\}. \end{aligned}$$

Therefore, $\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s\left(a_{2i-2}, a_{2i-1}\right), q_s\left(a_{2i-3}, a_{2i-2}\right)\right\} = \max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s(a_{2i-3}, a_{2i-}\right.$

$$\tau + F\left(q_s(a_{2i-1}, a_{2i-2})\right) \leq F\left(\mu_s\left(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s\left(a_{2i-3}, a_{2i-2}\right)\right\}\right)\right),$$

which implies that

$$q_s(a_{2i-1}, a_{2i-2}) < \mu_s(\max\left\{q_s\left(a_{2i-2}, a_{2i-3}\right), q_s(a_{2i-3}, a_{2i-2})\right\}),$$

$$\mu_s^2 q_s(a_{2i-1}, a_{2i-2}) < \mu_s^3(\max\left\{q_s(a_{2i-3}, a_{2i-2}), q_s(a_{2i-2}, a_{2i-3})\right\}). \tag{2.13}$$

Now, By (2.12) and (2.13), we have

$$\max\left\{\mu_s^2 q_s(a_{2i-2}, a_{2i-1}), \mu_s^2 q_s(a_{2i-1}, a_{2i-2})\right\}$$

$$< \mu_s^3(\max\{q_s(a_{2i-3},a_{2i-2}),q_s(a_{2i-2},a_{2i-3})\}). \tag{2.14}$$

By (2.14) and (2.11), we have

$$q_s(a_{2i},a_{2i+1}) \le \max\left\{\mu_s^3(q_s(a_{2i-3},a_{2i-2})),\mu_s^3(q_s(a_{2i-2},a_{2i-3}))\right\}. \tag{2.15}$$

Following the patterns of inequalities (2.7), (2.11) and (2.15), we have

$$q_s(a_{2i},a_{2i+1}) \le \max\left\{\mu_s^{2i}(q_s(a_0,a_1)),\mu_s^{2i}(q_s(a_1,a_0))\right\}.$$

As $j = 2i$, so

$$q_s(a_j,a_{j+1}) \le \max\left\{\mu_s^{j}(q_s(a_0,a_1)),\mu_s^{j}(q_s(a_1,a_0))\right\}. \tag{2.16}$$

Now, by using (2.3), we have

$$\tau + F\left(q_s(a_{2i+1},a_{2i})\right) \le \tau + F\left(H_{q_s}(Sa_{2i},Ta_{2i-1})\right)$$

$$\le \tau + \max\left\{F\left(H_{q_s}(Ta_{2i-1},Sa_{2i})\right),F\left(H_{q_s}(Sa_{2i},Ta_{2i-1})\right)\right\}$$

As $a_{2i-1},a_{2i} \in B_{q_s}(a_0,r)\cap\{TS(a_n)\}$, $\rho_s^*(a_{2i},Sa_{2i}) \ge s$, $\rho_s^*(Sa_{2i-1},a_{2i-1}) \ge s$ and $\max\{H_{q_s}(Ta_{2i-1},Sa_{2i}),H_{q_s}(Sa_{2i},Ta_{2i-1}),Q_s(a_{2i-1},a_{2i}),Q_s(a_{2i},a_{2i-1})\} > 0$, by (2.1), we have

$$\begin{aligned}
&\tau + F\left(q_s(a_{2i+1},a_{2i})\right)\\
\le\ & F(\mu_s\left(Q_s(a_{2i},a_{2i-1})\right))\\
=\ & F\left(\mu_s\left(\max\{q_s\left(a_{2i},a_{2i-1}\right),q_s\left(a_{2i},a_{2i+1}\right),q_s\left(a_{2i-1},a_{2i}\right)\}\right)\right).
\end{aligned}$$

By inequality (2.7), we have

$$\tau + F\left(q_s(a_{2i+1},a_{2i})\right) < F\left(\mu_s\left(\max\left\{q_s\left(a_{2i-1},a_{2i}\right),q_s\left(a_{2i},a_{2i-1}\right)\right\}\right)\right).$$

Now,

$$q_s(a_{2i+1},a_{2i}) < \max\left\{\mu_s\left(q_s(a_{2i-1},a_{2i})\right),\mu_s(q_s(a_{2i},a_{2i-1})\right\}. \tag{2.17}$$

Now, by (2.10) and (2.17), we have

$$q_s(a_{2i+1},a_{2i}) \le \max\left\{\mu_s^2(q_s(a_{2i-1},a_{2i-2})),\mu_s^2(q_s(a_{2i-2},a_{2i-1}))\right\}. \tag{2.18}$$

Now, by (2.14) and (2.18), we have

$$q_s(a_{2i+1},a_{2i}) \le \max\left\{\mu_s^3(q_s(a_{2i-3},a_{2i-2})),\mu_s^3(q_s(a_{2i-2},a_{2i-3}))\right\}. \tag{2.19}$$

Following the patterns of inequalities (2.17), (2.18) and (2.19), we have

$$q_s(a_{2i+1},a_{2i}) \le \max\left\{\mu_s^{2i}(q_s(a_0,a_1)),\mu_s^{2i}(q_s(a_1,a_0))\right\}.$$

As $j = 2i$, so

$$q_s(a_{j+1},a_j) \le \max\left\{\mu_s^{j}(q_s(a_0,a_1)),\mu_s^{j}(q_s(a_1,a_0))\right\}. \tag{2.20}$$

Now, if $j = 2i-1$, then inequalities (2.16) and (2.20) can be obtained by using similar arguments. Now, by using the triangle inequality, (2.16) and (2.2), we have

$$\begin{aligned} q_s(a_0, a_{j+1}) &\leq sq_s(a_0, a_1) + s^2 q_s(a_1, a_2) + ... + s^j q_s(a_{j-1}, a_j) + s^j q_s(a_j, a_{j+1}) \\ &\leq sq_s(a_0, a_1) + ... + s^j q_s(a_{j-1}, a_j) + s^{j+1} q_s(a_j, a_{j+1}) \\ &< sq_s(a_0, a_1) + s^2 \mu_s q_s(a_0, a_1) + ... + s^{j+1} \mu_s^j q_s(a_0, a_1) \\ &< \sum_{i=0}^{j} s^{i+1} \left[\max\left\{\mu_s^i(q_s(a_1, a_0)), \mu_s^i(q_s(a_0, a_1))\right\}\right] < r. \end{aligned}$$

Similarly, by using the triangle inequality, (2.20) and (2.2), we have

$$q_s(a_{j+1}, a_0) < \sum_{i=0}^{j} s^{i+1} \left[\max\left\{\mu_s^i(q_s(a_1, a_0)), \mu_s^i(q_s(a_0, a_1))\right\}\right] < r,$$

$$q_s(a_0, a_{j+1}) < r \text{ and } q_s(a_{j+1}, a_0) < r.$$

It follows that $a_{j+1} \in B_{q_s}(a_0, r)$. Also $\rho_s^*(Sa_{j+1}, a_{j+1}) \geq s$, $q_s(a_{j+1}, Ta_{j+1}) = q_s(a_{j+1}, a_{j+2})$ and $q_s(Ta_{j+1}, a_{j+1}) = q_s(a_{j+2}, a_{j+1})$, so from assumption (i), we have $\rho_s^*(a_{j+2}, Sa_{j+2}) \geq s$. Now, if $a_3, ...a_l \in B_{q_s}(a_0, r)$, and $\rho_s^*(a_0, Sa_0) \geq s$, $\rho_s^*(Sa_1, a_1) \geq s$, $\rho_s^*(Sa_3, a_3) \geq s$, $\cdots$, $\rho_s^*(a_{l+1}, Sa_{l+1}) \geq s$, for some $l \in \mathbb{N}$, where $l = 2i+1$, $i = 1, 2, 3, ..., \frac{l-1}{2}$, then similarly we obtain $a_{l+1} \in B_{q_s}(a_0, r)$ and $\rho_s^*(Sa_{l+2}, a_{l+2}) \geq s$. Hence by mathematical induction $a_n \in B_{q_s}(a_0, r)$, $\rho_s^*(a_{2n}, Sa_{2n}) \geq s$ and $\rho_s^*(Sa_{2n+1}, a_{2n+1}) \geq s$ for all $n \in \mathbb{N} \cup \{0\}$. Also, $a_{2n} \in G(S)$. Now inequalities (2.16) and (2.20) can be written as

$$q_s(a_n, a_{n+1}) < \max\left\{\mu_s^n(q_s(a_1, a_0)), \mu_s^n(q_s(a_0, a_1))\right\}, \tag{2.21}$$

$$q_s(a_{n+1}, a_n) < \max\left\{\mu_s^n(q_s(a_1, a_0)), \mu_s^n(q_s(a_0, a_1))\right\}, \tag{2.22}$$

for all $n \in \mathbb{N}$. As $\sum_{w=1}^{+\infty} s^w \mu_s^w(t) < +\infty$, the series

$$\sum_{w=1}^{+\infty} s^w \mu_s^w(\max\left\{\mu_s^{e-1}(q_s(a_1, a_0)), \mu_s^{e-1}(q_s(a_0, a_1))\right\})$$

converges for each $e \in \mathbb{N}$. As $s\mu_s(t) < t$, so

$$\begin{aligned} &s^{w+1} \mu_s^{w+1}(\max\left\{\mu_s^{e-1}(q_s(a_1, a_0)), \mu_s^{e-1}(q_s(a_0, a_1))\right\}) \\ < \ &s^w \mu_s^w(\max\left\{\mu_s^{e-1}(q_s(a_1, a_0)), \mu_s^{e-1}(q_s(a_0, a_1))\right\}), \text{ for all } w \in \mathbb{N}. \end{aligned}$$

So for fix $\varepsilon > 0$ there exists $k_1(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{j=1}^{+\infty} s^j \mu_s^j\left(\max\left\{\mu_s^{k_1(\varepsilon)-1}(q_s(a_1, a_0)), \mu_s^{k_1(\varepsilon)-1}(q_s(a_0, a_1))\right\}\right) < \varepsilon.$$

Let $m, k, p \in \mathbb{N}$ with $m > k > k_1(\varepsilon)$, then

$$\begin{aligned}
& q_s(a_k, a_m) = q_s(a_k, a_{k+p}) \\
\leq \;& sq_s(a_k, a_{k+1}) + s^2 q_s(a_{k+1}, a_{k+2}) + ... + s^p q_s(a_{k+p-1}, a_{k+p}) \\
< \;& s\mu_s^k (\max\{(q_s(a_1, a_0)), (q_s(a_0, a_1))\}) \\
& + s^2 \mu_s^{k+1} (\max\{(q_s(a_1, a_0)), (q_s(a_0, a_1))\}) \\
& + ... + s^p \mu_s^{k+p-1} (\max\{(q_s(a_1, a_0)), (q_s(a_0, a_1))\}) \\
= \;& s\mu_s \max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\} + \\
& s^2 \mu_s^2 \max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\} \\
& + ... + s^p \mu_s^p \max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\} \\
< \;& \sum_{j=1}^{+\infty} s^j \mu_s^j \left(\max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\}\right) \\
< \;& \sum_{j=1}^{+\infty} s^j \mu_s^j \left(\max\left\{\mu_s^{k_1(\varepsilon)-1}(q_s(a_1, a_0)), \mu_s^{k_1(\varepsilon)-1}(q_s(a_0, a_1))\right\}\right) < \varepsilon.
\end{aligned}$$

Thus we proved that $\{TS(a_n)\}$ is a left K - Cauchy sequence in (X, q). As (X, q) is left K sequentially complete, so $\{TS(a_n)\} \to a^* \in X$ and

$$\lim_{n\to\infty} q_s(a_n, a^*) = \lim_{n\to\infty} q_s(a^*, a_n) = 0. \tag{2.23}$$

As $\{a_{2n}\}$ is a subsequence of $\{TS(a_n)\}$, so $a_{2n} \to a^*$. Also, $\{a_{2n}\}$ is a sequence in $G(S)$ and $G(S)$ is closed, so $a^* \in G(S)$ and therefore

$$\rho_s^*(a^*, Sa^*) \geq s. \tag{2.24}$$

Now, we show that a^* is a fixed point for S. We claim that $q_s(a^*, Sa^*) = q_s(Sa^*, a^*) = 0$. On contrary, we assume that $q_s(a^*, Sa^*) > 0$. Now

$$q_s(a^*, Sa^*) \leq s(q_s(a^*, a_{2n+2}) + q_s(a_{2n+2}, Sa^*)). \tag{2.25}$$

Then there exists $n_0 \in \mathbb{N}$ such that $q_s(a_{2n+2}, Sa^*) > 0$ for all $n \geq n_0$. By Lemma 1.8, $0 < q_s(a_{2n+2}, Sa^*) \leq H_{q_s}(Ta_{2n+1}, Sa^*)$, so

$$\max\{H_{q_s}(Ta_{2n+1}, Sa^*), H_{q_s}(Sa^*, Ta_{2n+1}), Q_s(a_{2n+1}, a^*), Q_s(a^*, a_{2n+1}) > 0,$$

for all $n \geq n_0$. By Lemma 1.8, we get

$$\begin{aligned}
& \tau + F(q_s(a_{2n+2}, Sa^*)) \leq \tau + F(H_{q_s}(Ta_{2n+1}, Sa^*)) \\
\leq \;& \tau + \max\{F(H_{q_s}(Sa^*, Ta_{2n+1})), F(H_{q_s}(Ta_{2n+1}, Sa^*))\}.
\end{aligned}$$

By assumption, inequality (2.1) holds for a^*. Also $\rho_s^*(a^*, Sa^*) \geq s$ and $\rho_s^*(Sa_{2n+1}, a_{2n+1}) \geq s$, then by (2.1), we have

$$\tau + F(q_s(a_{2n+2}, Sa^*)) \leq F(\mu_s(Q_s(a^*, a_{2n+1}))).$$

Since F is strictly increasing, we have

$$q_s\left(a_{2n+2}, Sa^*\right) < \mu_s\left(Q_s(a^*, a_{2n+1})\right).$$

Taking limit as $n \to \infty$, on both sides of above inequality, we get

$$\lim_{n\to\infty} q_s\left(a_{2n+2}, Sa^*\right) < \lim_{n\to\infty} \mu_s\left(Q_s(a^*, a_{2n+1})\right). \tag{2.26}$$

Now,

$$\begin{aligned} Q_s(a^*, a_{2n+1}) &= \max\Big\{q_s\left(a^*, a_{2n+1}\right), q_s\left(a^*, Sa^*\right), \\ &\quad \frac{q_s\left(a^*, Sa^*\right) q_s\left(a^*, Ta_{2n+1}\right) + q_s\left(a_{2n+1}, Ta_{2n+1}\right) q_s\left(a_{2n+1}, Sa^*\right)}{q_s\left(a^*, Ta_{2n+1}\right) + q_s\left(a_{2n+1}, Sa^*\right)}\Big\} \\ &\le \max\Big\{q_s\left(a^*, a_{2n+1}\right), q_s\left(a^*, Sa^*\right), \\ &\quad \frac{q_s\left(a^*, Sa^*\right) q_s\left(a^*, a_{2n+2}\right) + q_s\left(a_{2n+1}, a_{2n+2}\right) q_s\left(a_{2n+1}, Sa^*\right)}{q_s\left(a^*, Ta_{2n+1}\right) + q_s\left(a_{2n+1}, Sa^*\right)}\Big\}. \end{aligned}$$

Taking limit as $n \to \infty$ on both side above inequality, we get

$$\lim_{n\to\infty}\left(Q_s(a^*, a_{2n+1})\right) \le q_s\left(a^*, Sa^*\right).$$

Now, inequality (2.26) implies

$$\lim_{n\to\infty} q_s\left(a_{2n+2}, Sa^*\right) < \mu_s\left(q_s\left(a^*, Sa^*\right)\right).$$

Taking limit as $n \to \infty$ on both sides of inequality (2.25) and using the above inequality, we have

$$q_s\left(a^*, Sa^*\right) < s\mu_s\left(q_s\left(a^*, Sa^*\right)\right).$$

As $s\mu_s(t) < t$, so our assumption is wrong and $q_s(a^*, Sa^*) = 0$. Now, assume that $q_s(Sa^*, a^*) > 0$, then there exists $n_1 \in \mathbb{N}$ such that $q_s(Sa^*, a_{2n+2}) > 0$ for all $n \ge n_1$. By Lemma 1.8, $0 < q_s(Sa^*, a_{2n+2}) \le H_{q_s}(Sa^*, Ta_{2n+1})$, so

$$\max\{H_{q_s}\left(Ta_{2n+1}, Sa^*\right), H_{q_s}(Sa^*, Ta_{2n+1}), Q_s(a_{2n+1}, a^*), Q_s(a^*, a_{2n+1}) > 0,$$

for all $n \ge n_1$. As inequality (2.1) holds for a^*, $\rho_s^*(a^*, Sa^*) \ge s$ and $\rho_s^*(Sa_{2n+1}, a_{2n+1}) \ge s$, by Lemma 1.8 and (2.1), we have

$$\tau + F\left(q_s\left(Sa^*, a_{2n+2}\right)\right) \le F(\mu_s\left(Q_s(a^*, a_{2n+1})\right)).$$

Since F is strictly increasing, we have

$$q_s\left(Sa^*, a_{2n+2}\right) < \mu_s\left(Q_s(a^*, a_{2n+1})\right).$$

Taking limit as $n \to \infty$, on both sides, we have

$$\begin{aligned} \lim_{n\to\infty} q_s\left(Sa^*, a_{2n+2}\right) &< \lim_{n\to\infty} \mu_s\left(Q_s(a^*, a_{2n+1})\right) \\ &\le q_s\left(a^*, Sa^*\right) = 0. \end{aligned}$$

This is a contradiction, so $q_s(Sa^*, a^*) = 0$. Hence $a^* \in Sa^*$. As $\rho_s^*(a^*, Sa^*) \geq s$ and $q_s(a^*, Sa^*) = q_s(Sa^*, a^*) = q_s(a^*, a^*)$, then assumption (i) implies that

$$\rho_s^*(Sa^*, a^*) \geq s.$$

Now, we show that a^* is a fixed point for T. We claim that $q_s(a^*, Ta^*) = q_s(Ta^*, a^*) = 0$. On contrary, we assume that $q_s(a^*, Ta^*) > 0$ then there exists $n_2 \in \mathbb{N}$ such that $q_s(a_{2n+1}, Ta^*) > 0$ for all $n \geq n_2$. By Lemma 1.8, $0 < q_s(a_{2n+1}, Ta^*) \leq H_{q_s}(Sa_{2n}, Ta^*)$, so

$$\max\{H_{q_s}(Sa_{2n}, Ta^*), H_{q_s}(Ta^*, Sa_{2n}), Q_s(a_{2n}, a^*), Q_s(a^*, a_{2n}) > 0,$$

for all $n \geq n_2$. Now By Lemma 1.8 and (2.1), we get

$$\begin{aligned} \tau + F(q_s(a_{2n+1}, Ta^*)) &\leq \tau + F(H_{q_s}(Sa_{2n}, Ta^*)) \\ &\leq \tau + \max\{F(H_{q_s}(Sa_{2n}, Ta^*)), F(H_{q_s}(Ta^*, Sa_{2n}))\}. \end{aligned}$$

As inequality (2.1) holds for a^*, $\rho_s^*(a_{2n}, Sa_{2n}) \geq s$ and $\rho_s^*(Sa^*, a^*) \geq s$, by (2.1), we have

$$\tau + F(q_s(a_{2n+1}, Ta^*)) \leq F(\mu_s(Q_s(a_{2n}, a^*))).$$

Since F is strictly increasing, we have

$$q_s(a_{2n+1}, Ta^*) < \mu_s(Q_s(a_{2n}, a^*)).$$

Taking limit $n \to \infty$, on both sides of above inequality, we get

$$\lim_{n\to\infty} q_s(a_{2n+1}, Ta^*) < \lim_{n\to\infty} \mu_s(Q_s(a_{2n}, a^*)) = 0.$$

This is a contradiction, so $q_s(a^*, Ta^*) = 0$. Now assume that $q_s(Ta^*, a^*) > 0$, then there exists $n_3 \in \mathbb{N}$ such that $q_s(Ta^*, a_{2n+1}) > 0$ for all $n \geq n_3$. By Lemma 1.8, $0 < q_s(Ta^*, a_{2n+1}) \leq H_{q_s}(Ta^*, Sa_{2n})$, so

$$\max\{H_{q_s}(Sa_{2n}, Ta^*), H_{q_s}(Ta^*, Sa_{2n}), Q_s(a_{2n}, a^*), Q_s(a^*, a_{2n}) > 0,$$

for all $n \geq n_3$. Following similar arguments as above, we obtain

$$\lim_{n\to\infty} q_s(Ta^*, a_{2n+1}) < 0.$$

So, $q_s(Ta^*, a^*) = 0$. Hence $a^* \in Ta^*$. Hence, the pair (S, T) has a common fixed point a^* in $B_{q_s}(a_0, r)$. □

Example 2.3. Let $X = [0, \infty)$. Define $q_s : X \times X \to [0, \infty)$ by $q_s(x, y) = (x + 2y)^2$, if $x \neq y$ and $q_s(x, y) = 0$ if $x = y$. Then (X, q_s) is a left (right) K -sequentially complate quasi b-metric with $s = 2$. Let $\mathcal{R}$ be the binary relation on X defined by

$$\begin{aligned} \mathcal{R} &= \left\{(x, \frac{x}{5}) : x \in \{0, 1, \frac{1}{25}, \frac{1}{625}, ...\}\right\} \\ &\cup \left\{(\frac{x}{5}, x) : x \in \{\frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\}\right\}. \end{aligned}$$

Consider μ_s, a function on $[0,\infty)$ defined by $\mu_s(t) = \frac{3t}{8}$. Define the pair of multivalued mappings $T, S : X \to P(X)$ be

$$Tx = \begin{cases} \left[\frac{x}{5}, \frac{x}{4}\right], & \text{if } x \in [0,1] \\ [4x^3, x^6+5], & \text{if } x \in (1,\infty) \end{cases}, \quad Sx = \begin{cases} \{\frac{x}{5}\}, & \text{if } x \in [0,1] \\ [x^4, x^7], & \text{if } x \in (1,\infty) \end{cases}.$$

Define $\rho : X \times X \to [0,\infty)$ as follows:

$$\rho(x,y) = \begin{cases} 2 & \text{if } (x,y) \in \mathcal{R} \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

$$\begin{aligned} A &= \{x : \rho_2^*(x, Sx) \geq 2\} = \left\{0, 1, \frac{1}{25}, \frac{1}{625}, ...\right\}. \\ B &= \{y : \rho_2^*(Sy, y) \geq 2\} = \left\{0, \frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\right\}. \end{aligned}$$

Let $x_0 = 1$ and $r = 49$, then $B_{q_s}(x_0, r) = [0, 3)$. Now,

$$\begin{aligned} G(S) &= \{x : \rho_2^*(x, Sx) \geq 2 \text{ and } x \in B_{q_s}(x_0, r)\} \\ &= \left\{0, 1, \frac{1}{25}, \frac{1}{625}, ...\right\}. \end{aligned}$$

Clearly $G(S)$ is closed and contains x_0. So, condition (ii) of Theorem 2.2 is satisfied. Now, as $\frac{1}{5^{n-1}} \in B_{q_s}(x_0, r)$, for all $n \in \mathbb{N}$

$$q_s(\frac{1}{5^{n-1}}, T\frac{1}{5^{n-1}}) = q_s(\frac{1}{5^{n-1}}, \frac{1}{5 \times 5^{n-1}}).$$

and

$$q_s(T\frac{1}{5^{n-1}}, \frac{1}{5^{n-1}}) = q_s([\frac{1}{5 \times 5^{n-1}}, \frac{1}{5^{n-1}}).$$

As $\rho^*\left(\frac{1}{5^{n-1}}, S\frac{1}{5^{n-1}}\right) \geq 2$, for all $n \in \{1,3,5,...\}$. So $\rho^*\left(S\frac{1}{5\times 5^{n-1}}, \frac{1}{5\times 5^{n-1}}\right) \geq 2$, for all $n \in \{1,3,5,...\}$. Also, $\rho^*\left(S\frac{1}{5^{n-1}}, \frac{1}{5^{n-1}}\right) \geq 2$, for all $n \in \{2,4,6,...\}$. Then $\rho^*\left(\frac{1}{5\times 5^{n-1}}, S\frac{1}{5\times 5^{n-1}}\right) \geq 2$, for all $n \in \{2,4,6,...\}$. Also, $0 \in B_{q_s}(x_0, r)$, $q_s(0, T0) = q_s(0,0)$, $q_s(T0, 0) = q_s(0,0)$. As $\rho^*(0, S0) \geq 2$ if and only if $\rho^*(S0, 0) \geq 2$. So, condition (i) of Theorem 2.2 is satisfied. Now, for all $x, y \in B_{q_s}(x_0, r) \cap \{XTx_n\}$ with $\rho_2^*(Sy, y) \geq 2$, $\rho_2^*(x, Sx) \geq 2$. In general, if $x, y \in B_{q_s}(x_0, r)$, $\rho^*(x, Sx) \geq s$ and $\rho^*(Sy, y) \geq s$, then

$$x = \frac{1}{5^{n-1}}, \ y = \frac{1}{5^{m-1}},$$

where n is positive odd integer and m is positive even integer. Define the function $F : R^+ \to R$ by $F(x) = \ln(x)$ and $\tau \in (0, \frac{1}{58})$. After some calculations, it can easily be proved that (S,T) is a $F - \mu_s - \rho_s^*$ contraction on open ball. Thus, all the conditions of Theorem 2.2 are satisfied. Moreover, T and S have a common fixed point 0.

Theorem 2.4. *Let (X,d) be a metric space and $S, T : X \to X$ be the self mappings, suppose the following conditions satisfy:*

(i) the set $G = \{a \in a : \rho(a, Sa) \geq 1\}$ is closed and non-empty,

(ii) there exists a function $\mu \in \Psi$ *such that for every* $(a,y) \in X \times X$, $\rho(a,Sa) \geq 1$, $\rho(Sy,y) \geq 1$ *implies* $d(Sa,Ty) \leq \mu(d(a,y))$,

(iii) for every $a \in X$, *we have* $\rho(a,Sa) \geq 1$ *implies* $\rho(Ta,STa) \geq 1$, *and* $\rho(Sa,a) \geq 1$ *implies* $\rho(STa,Ta) \geq 1$.

Then for any $a_0 \in G$, *the Picard sequence* $\{T^n a_0\}$ *converges to some* $a^* \in X$ *and* a^* *is a common fixed point of* T *and* S.

Remark 2.5. By taking non-empty proper subsets of $Q_s(a,y)$ instead of $Q_s(a,y)$ in Theorem 2.2, we can obtain six different new results.

Remark 2.6. Results in right K-sequentially complete quasi b-metric spaces can be obtained in a similar way.

3. Fixed point results for graphic $F-\mu_s-\rho_s^*$ contractions in quasi b-metric spaces

Consistent with Jachymski [28], let (X,q_s) be a quasi b-metric space and denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e, $E(G) \supseteq \triangle$. We assume G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph [28], by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length $m (m \in N)$ is a sequence $\{x_i\}_{i=0}^m$ of $m+1$ vertices such that $x_0 = x$, $x_m = y$ and $(x_{n-1},x_n) \in E(G)$ for $i = 1,...,m$. A graph G is connected if there is a path between any two vertices.

Definition 3.1. Let (X,q_s) be a quasi b-metric space endowed with a graph G and $S,T : X \to P(X)$ be multivalued mappings. The pair (S,T) is called $F-\mu_s-$ graphic contraction on the intersection of an open ball and a sequence if $\mu_s \in \Psi$, $F \in \Omega$, $x_0 \in X$, $r,\tau > 0$, $x,y \in B_{q_s}(x_0,r) \cap \{TS(x_n)\}$, $\{(x,v) \in E(G) : v \in Sx\}$ and $\{(u,y) \in E(G) : u \in Sy\}$, $q_s(x,Ty)+q_s(y,Sx) \neq 0$ and $\max\{H_{q_s}(Sx,Ty), H_{q_s}(Ty,Sx), Q_s(x,y), Q_s(y,x$
0, then

(i).

$$\tau + \max\{F(H_{q_s}(Sx,Ty)), F(H_{q_s}(Ty,Sx))\} \leq F(\mu_s(Q_s(x,y))), \tag{3.1}$$

and if $q_s(x,Ty)+q_s(y,Sx) = 0$, then

$$\max\{H_{q_s}(Sx,Ty), H_{q_s}(Ty,Sx), Q_s(x,y), Q_s(y,x)\} = 0.$$

(ii).

$$\sum_{i=0}^{j} s^{i+1}\left[\max\left\{\mu_s^i(q_s(x_1,x_0)), \mu_s^i(q_s(x_0,x_1))\right\}\right] < r, \text{ for all } j \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Theorem 3.2. *Let (X, q_s) be a left K sequentially complete quasi b-metric space endowed with graph G, $r > 0$, $x_0 \in B_{q_s}(x_0, r)$ and (S, T) be a $F - \mu_s-$ graphic contraction on the intersection of an open ball and a sequence. Suppose that the following assumptions hold:*

(i) If $x \in B_{q_s}(x_0, r)$, (a) $\{(x, v) \in E(G) : v \in Sx\}$, $q_s(x, Sx) = q_s(x, y)$ and $q_s(Sx, x) = q_s(y, x)$, then $\{(u, y) \in E(G) : u \in Sy\}$.

(b) $\{(v, x) \in E(G) : v \in Sx\}$, $q_s(x, Tx) = q_s(x, y)$ and $q_s(Tx, x) = q_s(y, x)$, then $\{(y, u) \in E(G) : u \in Sy\}$.

(ii) The set $A(S) = \{x : (x, v) \in E(G)$ for all $v \in Sx$ and $x \in B_{q_s}(x_0, r)\}$ is closed and contained x_0.

Then the subsequence $\{x_{2n}\}$ of $\{TS(x_n)\}$ is a sequence in $G(S)$ and $\{x_{2n}\} \to x^* \in G(S)$. Also, if inequality (3.1) holds for x^*. Then T and S have a common fixed point x^* in $B_{q_s}(x_0, r)$.

Proof. Define $\rho : X \times X \to [0, \infty)$ by $\rho(x, v) = s$, for all $v \in Sx$, and $x \in B_{q_s}(x_0, r) \cap \{TS(x_n)\}$ with $\{(x, v) \in E(G) : v \in Sx\}$. Also $\rho(u, y) = s$, for all $u \in Sy$, and $y \in B_{q_s}(x_0, r) \cap \{TS(x_n)\}$ with $\{(u, y) \in E(G) : u \in Sy\}$. Moreover $\rho(x, y) = 0$, otherwise. Now, as (S, T) is a $F - \mu_s-$ graphic contraction. So inequality (3.1) implies inequality (2.1). Inequality (3.2) implies inequality (2.2). Assumption (i) of Theorem 3.2 implies assumption (i) of Theorem 2.2 and assumption (ii) of Theorem 3.2 implies assumption (ii) of Theorem 2.2. So, all conditions of Theorem 2.2 are satisfied. Hence the subsequence $\{x_{2n}\}$ of $\{TS(x_n)\}$ is a sequence in $A(S)$, for all $n \in N \cup \{0\}$ and $\{x_{2n}\} \to x^* \in A(s)$. Also, if inequality (3.1) holds for x^*, then inequality (2.1) holds for x^*. Hence all conditions of Theorem 2.2 are satisfied. Then T and S have a common fixed point x^* in $B_{q_s}(x_0, r)$. □

Theorem 3.3. *Let (X, d) be a complete metric space endowed with graph G and S,$T : X \to X$ be two self mappings. Suppose that the following assumptions hold:*

(i) there exists a function $\mu \in \Psi$ such that for every $(x, y) \in X \times X$, $(x, Sx) \in E(G)$, $(Sy, y) \in E(G) \Rightarrow d(Sx, Ty) \leq \mu(d(x, y))$

(ii) if $(x, Sx) \in E(G)$, then $(Tx, STx) \in E(G)$ and if $(Sx, x) \in E(G)$, then $(STx, Tx) \in E(G)$.

(iii) The set $G(S) = \{x : (x, Sx) \in E(G)\}$ is closed and non-empty.

Then T and S have a common fixed point x^* in X.

Ran and Reurings [39] extended the Banach contraction principle in partially ordered sets. We will deduce very easily various fixed point results on a complete left (right) K sequentially quasi b-metric space endowed with a partial order. Now, we have the following results.

Theorem 3.4. *Let $(X,\preceq)$ be a partial order set and (X,q_s) be a complete left (right) K sequentially quasi b-metric space. Let $r>0$, $x_0\in X$ and $S,T:X\to P(X)$ be the mappings on $B_{q_s}(x_0,r)$ and suppose there exist a function $\mu_s\in\Psi$ and a strictly increasing mapping F satisfying the following:*

(i). *for all $(x,y)\in B_{q_s}(x_0,r)\cap\{TS(x_n)\}$ with $Sy\preceq y$ and $x\preceq Sx$, if $\max\{H_{q_s}(Sx,Ty),H_{q_s}(Ty,Sx),Q$ 0 and $q_s(x,Ty)+q_s(y,Sx)\neq 0$, we have*

$$\tau+\max\{F\left(H_{q_s}\left(Sx,Ty\right)\right),F\left(H_{q_s}\left(Ty,Sx\right)\right)\}<F\left(\mu_s\left(Q_s\left(x,y\right)\right)\right)$$

where,

$$Q_s\left(x,y\right)=\max\left\{q_s\left(x,y\right),q_s\left(x,Sx\right),\frac{q_s\left(x,Sx\right)q_s\left(x,Ty\right)+q_s\left(y,Ty\right)q_s\left(y,Sx\right)}{q_s(x,Ty)+q_s(y,Sx)}\right\}.$$

If $q_s(x,Ty)+q_s(y,Sx)=0$, then

$$\max\{H_{q_s}\left(Sx,Ty\right),H_{q_s}\left(Ty,Sx\right),Q_s(x,y),Q_s(y,x)\}=0.$$

(ii). $\sum_{i=0}^{j}s^{i+1}\left[\max\left\{\mu^i q_s\left(x_1,x_0\right),\mu^i q_s\left(x_0,x_1\right)\right\}\right]<r$, *for all $j\in N\cup\{0\}$.*

(iii). *If $x\in B_{q_s}(x_0,r)$, (a) $x\preceq Sx$, $q_s\left(x,Sx\right)=q_s\left(x,y\right)$ and $q_s\left(Sx,x\right)=q_s\left(y,x\right)$ implies $Sy\preceq y$.*

(b) $Sx\preceq x$, $q_s\left(x,Tx\right)=q_s\left(x,y\right)$ and $q_s\left(Tx,x\right)=q_s\left(y,x\right)$ implies $y\preceq Sy$.

(iv). *The set $G(S)=\{x:x\preceq Sx$ and $x\in B_{q_s}(x_0,r)\}$ is closed and contained x_0.*

Then the subsequence $\{x_{2n}\}$ of $\{TS\left(x_n\right)\}$ is a sequence in $G\left(S\right)$ and $\{x_{2n}\}\to x^*\in G(S)$. Also, if assumption (i) holds for x^*. Then T and S have a common fixed point x^* in $B_{q_s}(x_0,r)$.

Proof. Define $\rho:X\times X\to[0,\infty)$, by $\rho\left(x,v\right)=s$, for all $v\in Sx$, where $x\in B_{q_s}(x_0,r)\cap\{TS\left(x_n\right)\}$ with $x\preceq Sx$. Also $\rho\left(u,y\right)=s$, for all $u\in Sy$, where $y\in B_{q_s}(x_0,r)\cap\{TS\left(x_n\right)\}$ with $y\succeq Sy$. Moreover $\rho\left(x,y\right)=0$, otherwise. It is easy to see that assumptions (i), (ii), (iii) and (iv) of Theorem 3.4 imply inequality (2.1), inequality (2.2), assumption (i) and assumption (ii) of Theorem 2.2 respectively. So, all conditions of Theorem 2.2 are satisfied. Hence the subsequence $\{x_{2n}\}$ of $\{TS\left(x_n\right)\}$ is a sequence in $G\left(S\right)$, for all $n\in N\cup\{0\}$ and a sequence $\{x_{2n}\}\to x^*\in G(s)$. Also, if assumption (i) holds for x^*, then inequality (2.1) holds for x^*. Then T and S have a common fixed point x^* in $B_{q_s}(x_0,r)$. □

Theorem 3.5. ***Theorem 3.5*** *Let $(X,\preceq,d)$ be an ordered metric space and $S,T:X\to X$ be the self mappings, suppose the following conditions hold:*

(i) *the set $G=\{x\in X:x\preceq Sx\}$ is closed and non-empty,*

(ii) *there exists a function $\mu\in\Psi$ such that for every $(x,y)\in X\times X$, $x\preceq Sx$, $y\succeq Sy$ $\Rightarrow d\left(Sx,Ty\right)\leq\mu\left(d\left(x,y\right)\right)$,*

(iii) *for every $x\in X$, we have $x\preceq Sx\Rightarrow Tx\succeq STx$, $x\succeq Sx\Rightarrow Tx\preceq STx$.*

Then for any $x_0 \in G$, the Picard sequence $\{T^n x_0\}$ converges to some $x^* \in X$ and x^* is a common fixed point.

4. **Application to system of integral equations**

Let $S, T : X \to X$ be two self mappings and $x_0 \in X$. Let $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$ and so on. In this way, we construct a sequence $\{x_n\}$ in X such that

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ (where } i = 0, 1, 2, \ldots).$$

We say that $\{TS(x_n)\}$ is a sequence in X generated by x_0.

Definition 4.1. Let (X, q_s) be a left (right) K -sequentially complete quasi b-metric space and $S, T : X \to X$ be two self mappings. The pair (S, T) is called a $F - \mu_s$ contraction, if there exist $F \in \mathcal{F}_Q, \tau, a > 0, x, y \in X, \max\{q_s(Sx, Ty), q_s(Ty, Sx), Q_s(x, y), Q_s(x, y)\} > 0$ and $q_s(x, Ty) + q_s(y, Sx) \neq 0$, then

$$\tau + \max\{F(q_s(Sx, Ty)), F(q_s(Ty, Sx))\} \leq F(\mu_s(Q_s(x, y))), \tag{4.1}$$

and if $q_s(x, Ty) + q_s(y, Sx) = 0$, then

$$\max\{q_s(Sx, Ty), q_s(Ty, Sx), Q_s(x, y), Q_s(x, y)\} > 0,$$

where

$$Q_s(x, y) = \max\left\{q_s(x, y), q_s(x, Sx), \frac{q_s(x, Sx)\, q_s(x, Ty) + q_s(y, Ty)\, q_s(y, Sx)}{q_s(x, Ty) + q_s(y, Sx)}\right\}. \tag{4.2}$$

Then we deduce the following main result.

Theorem 4.2. *Let (X, q_s) be a left (right) K-sequentially complete quasi b metric space with constant $s \geq 1$ and (S, T) be a $F - \mu_s$ contraction. Then $\{TS(x_n) \to x^* \in X$. Also, if x^* satisfies (4.1), then S and T have a unique common fixed point x^* in X.*

Proof. Now, we have to prove uniqueness only. Let u be another common fixed point of S, T. If $\max\{q_s(Su, Tx^*), q_s(Tx^*, Su), Q_s(x^*, u), Q_s(u, x^*)\} \not> 0$, or if $q_s(x^*, Tu) + q_s(u, Sx^*) = 0$, then $q_s(Su, Tx^*) = 0$ and $q_s(Tx^*, Su) = 0$, which further implies $q_s(u, x^*) = q_s(x^*, u) = 0$ and hence $u = x^*$. Now, suppose $q_s(x^*, u) > 0$, then $\max\{q_s(Su, Tx^*), q_s(Tx^*, Su), Q_s(x^*, u), Q_s(u, x^*)\} > 0$ and $q_s(x^*, Tu) + q_s(u, Sx^*) \neq 0$. Then, we have

$$\begin{aligned} \tau + F(q_s(Su, Tx^*)) &\leq \tau + \max\{F(q_s(Su, Tx^*)), F(q_s(Tx^*, Su))\} \\ &\leq F(\mu_s(Q_s(x^*, u))). \end{aligned}$$

This implies that

$$q_s(u, x^*) < \mu_s(q_s(u, x^*)) < s\mu_s(q_s(u, x^*))$$

which is contradiction. Then, we get $q_s(u, x^*) = 0$. Similarly we obtain $q_s(x^*, u) = 0$. Hence $x^* = u$. □

Now, as an application, we discuss the application of Theorem 4.2 to find solution of the system of Volterra type integral equations. Consider the following integral equations:

$$u(t) = \int_0^t K_1(t, s, u(s))ds, \tag{4.3}$$

$$v(t) = \int_0^t K_2(t, s, v(s))ds \tag{4.4}$$

for all $t \in [0, 1]$. We find the solution of (4.3) and (4.4). Let $X = C([0, 1], \mathbb{R}_+)$ be the set of all continuous functions on $[0, 1]$, endowed with the complete a left (right) K-sequentially quasi b-metric. For $u \in C([0, 1], \mathbb{R}_+)$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,1]} \{(u(t))\, e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$q_\tau(u, v) = \left[\sup_{t\in[0,1]} \{(u(t) + 2v(t))e^{-\tau t}\}\right]^2 = \|u + 2v\|_\tau^2$$

for all $u, v \in C([0, 1], \mathbb{R}_+)$, with these settings, $(C([0, 1], \mathbb{R}_+), q_\tau)$ becomes a quasi b-metric space.

Now we prove the following theorem to ensure the existence of solution of integral equations.

Theorem 4.3. *Assume that* $K_1, K_2 : [0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$. *Define*

$$\begin{aligned} Su(t) &= \int_0^t K_1(t, s, u(s))ds, \\ Tv(t) &= \int_0^t K_2(t, s, v(s))ds. \end{aligned}$$

Suppose there exists $\tau > 0$, *such that*

$$\max\{K_1(t, s, u) + 2K_2(t, s, v), K_2(t, s, v) + 2K_1(t, s, u)\} < \frac{\tau M(u, v)e^{\tau s}}{\tau M(u, v) + 1} \tag{4.5}$$

for all $t, s \in [0, 1]$ *and* $u, v \in C([0, 1], \mathbb{R})$, *where*

$$M(u, v) = \mu_s\left(\max\left\{\begin{array}{c} \|u + 2v\|^2, \|u + 2Su\|^2, \\ \frac{\|u+2Su\|^2\|u+2Tv\|^2+\|v+2Tv\|^2\|v+2Su\|^2}{\|u+2Tv\|^2+\|v+2Su\|^2} \end{array}\right\}\right).$$

Then integral equations (4.3) and (4.4) have a unique common solution.

Proof. By assumption (ii)

$$\begin{aligned}
& |\max\{Su+2Tv, Tv+2Su\}| \\
= \; & \max\left\{\int_0^t (K_1(t,s,u)+2K_2(t,s,v))ds, \int_0^t (K_2(t,s,v)+2K_1(t,s,u))ds\right\} \\
< \; & \int_0^t \frac{\tau M(u,v)}{\tau M(u,v)+1} e^{\tau s} ds \\
< \; & \frac{\tau M(u,v)}{\tau M(u,v)+1} \int_0^t e^{\tau s} ds,
\end{aligned}$$

$$\begin{aligned}
|\max\{Su+2Tv, Tv+2Su\}| &< \frac{\tau M(u,v)(e^{\tau t}-1)}{(\tau M(u,v)+1)\tau}, \\
&< \frac{M(u,v)e^{\tau t}}{\tau M(u,v)+1}, \\
|\max\{Su+2Tv, Tv+2Su\}| \, e^{-\tau t} &< \frac{M(u,v)}{\tau M(u,v)+1}, \\
||\max\{Su+2Tv, Tv+2Su\}||_\tau &< \frac{M(u,v)}{\tau M(u,v)+1}.
\end{aligned}$$

This implies

$$\frac{\tau M(u,v)+1}{M(u,v)} < \frac{1}{||\max\{Su+2Tv, Tv+2Su\}||_\tau}.$$

That is

$$\tau + \frac{1}{M(u,v)} < \frac{1}{||\max\{Su+2Tv, Tv+2Su\}||_\tau},$$

which further implies

$$\begin{aligned}
\tau - \frac{1}{||\max\{Su+2Tv, Tv+2Su\}||_\tau} &< \frac{-1}{M(u,v)}, \\
\tau + \max\{\frac{-1}{||Su+2Tv||_\tau}, \frac{-1}{||Tu+2Sv||_\tau}\} &< \frac{-1}{M(u,v)}.
\end{aligned}$$

So all the conditions of Theorem 4.3 are satisfied for $F(v) = \frac{-1}{\sqrt{v}}$, $v > 0$ and $q_\tau(u,v) = \|u+2v\|_\tau^2$. Hence integral equations given in (4.3) and (4.4) have a unique common solution. □

5. Application to functional equations

In this section, we derive an application for the solution of a functional equation arising in dynamic programming. Consider U and V two Banach spaces, $P \subseteq U$,

$Q \subseteq V$ and

$$\begin{aligned} f &: P \times Q \to P \\ g, u &: P \times Q \to \mathbb{R} \\ M, N &: P \times Q \times \mathbb{R} \to \mathbb{R}. \end{aligned}$$

For further results on dynamic programming, we refer to [7, 18, 19, 38]. Suppose that P and Q represent the state and decision spaces, respectively. The problem related to dynamic programming is reduced to solve the following functional equations:

$$p(\gamma) = \sup_{\alpha \in Q} \{ g(\gamma, \alpha) + M(\gamma, \alpha, p(f(\gamma, \alpha))) \}, \tag{5.1}$$

$$q(\gamma) = \sup_{\alpha \in Q} \{ u(\gamma, \alpha) + N(\gamma, \alpha, q(f(\gamma, \alpha))) \}, \tag{5.2}$$

for $\gamma \in P$. We ensure the existence and uniqueness of a common and bounded solution of Equations (5.1) and (5.2). Suppose $B(P)$ is the set of all bounded real valued functions on P. Consider,

$$d_s(h, k) = \|h - k\|_\infty^2 = \sup_{\gamma \in P} |h(\gamma) - k(\gamma)|^2 \tag{5.3}$$

for all $h, k \in B(P)$. Then $(B(P), d_s)$ is a quasi b-metric space. Assume that
$(C1)$: M, N, g, and u are bounded.
$(C2)$: For $\gamma \in P$, $h \in B(P)$, $S, T : B(P) \to B(P)$, take

$$Sh(\gamma) = \sup_{\alpha \in Q} \{ g(\gamma, \alpha) + M(\gamma, \alpha, h(f(\gamma, \alpha))) \}, \tag{5.4}$$

$$Th(\gamma) = \sup_{\alpha \in Q} \{ u(\gamma, \alpha) + N(\gamma, \alpha, h(f(\gamma, \alpha))) \}. \tag{5.5}$$

Moreover, for every $(\gamma, \alpha) \in P \times Q$, $h, k \in B(P)$, $t \in P$ and $\tau > 0$,

$$|M(\gamma, \alpha, h(t)) - N(\gamma, \alpha, k(t)| \leq D(h, k) e^{-\tau} \tag{5.6}$$

where,

$$D(h, k) = \mu_s \left(\max \left\{ \begin{array}{c} |h(t) - k(t)|^2, |h(t) - Sh(t)|^2, \\ \frac{|h(t) - Sh(t)|^2 |h(t) - Tk(t)|^2 + |k(t) - Tk(t)|^2 |k(t) - Sh(t)|^2}{|h(t) - Tk(t)|^2 + |k(t) - Sh(t)|^2} \end{array} \right\} \right).$$

Theorem 5.1. *Assume that the conditions $(C1)$, $(C2)$ and (5.6) hold. Then Equations (5.1) and (5.2) have a unique common and bounded solution in $B(P)$.*

Proof. Take any $\lambda > 0$. By using definition of supremum in equation (5.4) and (5.5), there exist $h_1, h_2 \in B(P)$, and $\alpha_1, \alpha_2 \in Q$ such that

$$(Sh_1) < g(\gamma, \alpha_1) + M(\gamma, \alpha_1, h_1(f(\gamma, \alpha_1))) + \lambda, \tag{5.7}$$

$$(Th_2) < g(\gamma, \alpha_2) + N(\gamma, \alpha_2, h_2(f(\gamma, \alpha_2))) + \lambda. \tag{5.8}$$

Again using definition of supremum, we have

$$(Sh_1) \geq g(\gamma, \alpha_2) + M(\gamma, \alpha_2, h_1(f(\gamma, \alpha_2))), \tag{5.9}$$

$$(Th_2) \geq g(\gamma, \alpha_1) + N(\gamma, \alpha_1, h_2(f(\gamma, \alpha_1))). \tag{5.10}$$

Then equations (5.7), (5.10) and (5.6) imply

$$\begin{aligned} & (Sh_1)(\gamma) - (Th_2)(\gamma) \\ \leq \; & M(\gamma, \alpha_1, h_1(f(\gamma, \alpha_1))) - N(\gamma, \alpha_1, h_2(f(\gamma, \alpha_1))) + \lambda \\ \leq \; & |M(\gamma, \alpha_1, h_1(f(\gamma, \alpha_1))) - N(\gamma, \alpha_1, h_2(f(\gamma, \alpha_1)))| + \lambda \\ \leq \; & D(h, k)e^{-\tau} + \lambda. \end{aligned}$$

Since, $\lambda > 0$ is arbitrary, we get

$$\begin{aligned} |Sh_1(\gamma) - Th_2(\gamma)| &\leq D(h, k)e^{-\tau} \\ e^{\tau} |Sh_1(\gamma) - Th_2(\gamma)| &\leq D(h, k). \end{aligned}$$

This further implies,

$$\tau + \ln |Sh_1(\gamma) - Th_2(\gamma)| \leq \ln(D(h, k).$$

Therefore, all requirements of Theorem 4.1 hold for $F(g) = \ln g$; $g > 0$ and $d_\tau(h, k) = \|h - k\|_\tau^2$. Thus, there exists a common fixed point $h^* \in B(W)$ of S and T, that is, $h^*(\gamma)$ is a unique common solution of equations (5.1) and (5.2). □

6. **Conclusion**

In the present paper, we have introduced $F - \mu_s - \rho_s^*$ contractive condition on a sequence contained in an open ball to ensure the existence of a fixed point for a pair of multivalued mappings. A weaker class Ω of strictly increasing mappings is used rather than the class of mappings used by Wardowski [49]. Example is given to demonstrate the variety of our results. Fixed point results with graphic contractions for a pair of multivalued mappings are established. Results endowed with a partial order have been obtained. Two different types of applications on Voltera type nonlinear integral inclusions and dynamical process are presented. Moreover, we investigate our results in a better framework of quasi b-metric spaces. New results in ordered spaces, b -metric space, quasi metric space and metric space can be obtained as corollaries of our results.

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