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Construction of similarity transformations and analytic solutions for a liquid film on an unsteady stretching sheet using lie point symmetries

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ABSTRACT

We obtain Lie point symmetries for the system of partial differential equations describing flow and heat transfer in a thin liquid film on an unsteady stretching sheet and use them to construct invariants. We derive similarity transformations using deduced invariants that reduce the independent variables of the considered flow model. Such reductions lead to systems of ordinary differential equations. We solve these systems of ordinary differential equations analytically by applying the Homotopy analysis method.

1. Introduction

Industrial applications of the flow and heat transfer in a thin liquid film have stimulated a great deal of research over the past several decades. Substantial effort has been expended in developing approximation and analytic techniques. By applying such methods solutions of the flow problems are approximated that help in analyzing, for example, the hydrodynamics of a flow in a thin liquid film caused by an unsteady stretching surface in [1] that is further extended to include heat transfer analysis [2]. An analytic investigation of thin film flow of biviscosity liquid over an unsteady stretching sheet is carried out in [3] where a quick thinning of the film is shown for biviscosity fluids as compared to Newtonian fluids. Analytic solutions of such problems are constructed by employing the Homotopy analysis method (HAM) that provides a series solution [4], and are reported in [5,6]. In the study of thin film flow over an unsteady stretching surface, the effects of the magnetic field and viscous dissipation with general surface temperature are included in [7,8], thermo capillary and magnetic field in [9] and internal heat generation with a general surface temperature in [10]. The study of boundary layer flow of nanofluids has emerged as another stream of research [11]. These fluids contain nanoparticles like metal, carbides, nitrides, and oxides, etc. Such nano sized particles are suspended in a base fluid to engineer nanofluids. The presence of nanoparticles is reported to enhance the thermal conductivity and heat transfer characteristics [12].

The flow and heat transfer in a thin fluid film on an unsteady stretching surface has considerable effects on many industrial products like refinement and quality improvement. The thermal and momentum flow in such processes have been analyzed extensively, by applying numerical and perturbation techniques. Analytic solutions for this class of problems have also been constructed using HAM, for example, in [13] this procedure is adopted and a good agreement with the results derived numerically [2] is reported. In [13] the fluid flow and heat transfer are considered in a thin Newtonian film of uniform thickness. The flow is caused due to stretching of the horizontal elastic sheet. Further, variable surface temperature (varying with distance and time) is considered along with the motion of the sheet with a prescribed velocity. The boundary layer equations describing the velocity and temperature fields in the thin liquid film are second order partial differential equations (PDEs) with three dependent and three independent variables. A similarity transformation is applied to map them to a coupled system of ordinary differential equations (ODEs). An analytic solution for the deduced system of ODEs is presented using HAM.

Lie symmetry method is an algebraic technique that in most cases provides invariant analytic solutions for those differential equations (DEs) which admit some Lie point symmetries, see e.g [14–19]. Lie symmetry method has been employed to study, e.g. the hyperbolic shallow water equations and the Green Naghdi model [20], shallow water equations in the Boussinesq approxi-

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mation [21], shallow water equations with complete Coriolis force [22], rotating shallow water [23], shallow water equations on a rotating plane [24], two-Dimensional shallow water equations with constant coriolis parameter [25]. Moreover, one dimensional optimal system is constructed to reduce partial differential equations describing the two-dimensional rotating ideal gas, to ordinary differential equations that were further solved by quadratures [26]. Approximate symmetries have also been employed to derive similarity solutions for wave equations on liquid films [27]. Invariant solutions for a DE remain unaltered under the action of the symmetry group spanned by the associated Lie point symmetry generators. A Lie point symmetry of a system of PDEs offers reduction through associated invariants (similarity transformations) in the number of its independent and dependent variables. We are interested in reducing the independent variables as such successive reductions lead to a system of ODEs for which there exist well established numerical and analytic solution schemes. In this paper we find a Lie algebra that is spanned by seven Lie point symmetry generators for the system of PDEs with three dependent and three independent variables, describing the flow and heat transfer in a thin fluid film on an unsteady stretching surface. Further, we deduce invariants with the help of obtained symmetries which transform the considered model to a system of PDEs with two independent variables, when we go for a single reduction through these invariants. Further, with symmetries of the once reduced systems of PDEs, we derive associated invariants that map them to systems of ODEs. To achieve such double reductions of the concerned model we present an explicit procedure to assemble similarity transformations in each case with the help of two sets of invariants drawn from symmetries of the model and its first reduction. For the resulting systems of ODEs, we construct series solutions by employing HAM. Lie point symmetry generators reveal many solvable classes here to study and analyze the flow and heat transfer in a thin liquid film. However, they dictate the form of film thickness, velocity, and temperature of the stretching sheet, to consider initially with the flow model. In [13] specific forms of these three are considered to facilitate the construction of the similarity transformations, while here they are determined through the admitted Lie algebra.

The outline of the paper is as follows. The second section is on review of the considered model, derivation of Lie point symmetries and invariants. The third section is on the construction of the similarity transformations and double reductions of the considered model. In the fourth section analytic solutions are presented for systems of ODEs derived through double reductions. The last section is on conclusion and discussion.

2. Lie symmetries and invariants for the flow model

A Newtonian fluid flow in a thin liquid film on a horizontal elastic sheet analyzed in [13]. The film has uniform thickness $h(t)$ while the flow within the film is due to the stretching of the sheet. This work further considered the smoothness of the surface of the planar liquid film with no surface waves. Moreover, the viscous sheer stress along with heat flux at the adiabatic free surface vanishes. Under the said assumptions the following two dimensional boundary layer equations are presented in [13] to describe the velocity and temperature fields in the thin liquid film

$$\begin{aligned} &u_x+v_y=0,\\ &u_t+uu_x+vu_y-\nu u_{yy}=0,\\ &T_t+uT_x+vT_y-\kappa T_{yy}=0, \end{aligned} \tag{1}$$

where u and v are the x and y-components of velocity and T is the temperature of the fluid. The subscripts in the above equations denote partial derivatives, e.g. time t in the subscript is a partial derivative $u_t=\frac{\partial u}{\partial t}$. Further, ν is the kinematic viscosity, κ is thermal diffusivity. The conditions that are associated with the above system read as

$$\begin{aligned} &y=0,\quad u=U(x,t),\quad v=0,\quad T=T_s(x,t),\\ &y=h(t),\quad u_y=T_y=0,\quad v=h_t=\frac{dh}{dt}. \end{aligned} \tag{2}$$

The second set of conditions imposes a kinematic constraint of the fluid motion. The stretching surface velocity considered is

$$U=\frac{bx}{(1-\alpha t)}, \tag{3}$$

that causes the flow in this problem and the temperature of the elastic sheet is taken as

$$T_s=T_0-T_{ref}\frac{bx^2}{2\nu}(1-\alpha t)^{\frac{-3}{2}}, \tag{4}$$

where T_0 and T_{ref} are temperature at the slit and reference temperature for all $t<\frac{1}{\alpha}$. In these expressions b and α are positive constants and these specific forms of the surface velocity and temperature of the stretching sheet are chosen to facilitate the construction of the following similarity transformation

$$\begin{aligned} &\eta=\sqrt{\frac{b}{\nu(1-\alpha t)}}\frac{y}{\beta},\ u=\frac{bx}{1-\alpha t}f'(\eta),\ v=-\beta\sqrt{\frac{b\nu}{(1-\alpha t)}}f(\eta),\\ &T=T_0-T_{ref}\frac{bx^2}{2\nu(1-\alpha t)^{\frac{3}{2}}}\theta(\eta). \end{aligned} \tag{5}$$

This transformation satisfies the continuity equation and setting $\eta=1$ at the free surface yields $h(t)=\beta\sqrt{\frac{\nu(1-\alpha t)}{b}}$ and maps (1)-(2) into the following system of ODEs

$$\begin{aligned} &f'''+\beta^2\left((f-\frac{S\eta}{2})f''-(f'+S)f'\right)=0,\\ &Pr^{-1}\theta''+\beta^2\left((f-\frac{S\eta}{2})\theta'-(2f'+\frac{3S}{2})\theta\right)=0, \end{aligned} \tag{6}$$

and conditions

$$\begin{aligned} &f(0)=0,\ f'(0)=1,\ \theta(0)=1,\ f(1)=S/2,\ f''(1)=0,\\ &\theta'(1)=0, \end{aligned} \tag{7}$$

where prime denotes derivative with respect to η, $Pr=\frac{\nu}{\kappa}$ is the Prandtl number, β is a constant determined during construction of the analytic solution using HAM, $S=\frac{\alpha}{b}$ is dimensionless measure of the unsteadiness.

We derive Lie point symmetry generators for system (1) to deduce similarity transformations of the form (5) to transform it into systems of ODEs. A lie point symmetry for (1) is a vector field

$$\mathbf{X}=\xi_l\frac{\partial}{\partial\psi_l}+\eta_l\frac{\partial}{\partial\zeta_l},\qquad l=1,2,3, \tag{8}$$

where summation is over repeated indices l. Infinitesimal coordinates are denoted by ξ_l, η_l that are functions of the independent variables t,x,y and the dependent variables u,v,T. While ψ_l and ζ_l denote the independent and dependent variables, i.e., $\psi_1=t$, $\psi_2=x$, $\psi_3=y$, $\zeta_1=u$, $\zeta_2=v$ and $\zeta_3=T$. The concerned system (1) is of second order PDEs while a couple of associated conditions (2) and the continuity equations are written in terms of first order partial derivatives, therefore we need second $\mathbf{X}^{[2]}$ and first extensions $\mathbf{X}^{[1]}$, of (8) to operate on them. To demonstrate extension procedure we consider the following formula

$$\begin{aligned} \mathbf{X}^{[2]}=\mathbf{X}&+\eta_l^t\frac{\partial}{\partial\zeta_{l,t}}+\eta_l^x\frac{\partial}{\partial\zeta_{l,x}}+\eta_l^y\frac{\partial}{\partial\zeta_{l,y}}\\ &+\eta_l^{tt}\frac{\partial}{\partial\zeta_{l,tt}}+\eta_l^{xx}\frac{\partial}{\partial\zeta_{l,xx}}+\eta_l^{yy}\frac{\partial}{\partial\zeta_{l,yy}}, \end{aligned} \tag{9}$$

that also contains all the first extension components. The first and second extension coefficients are obtainable from the following formulas

$$\eta_l^m = D_m\eta_l - \zeta_{l,t}D_m(\xi_1) - \zeta_{l,x}D_m(\xi_2) - \zeta_{l,y}D_m(\xi_3),$$
$$m \in \{t, x, y\},\ \ l = 1, 2, 3, \tag{10}$$

and

$$\eta_l^{tt} = D_t\eta_l^t - \zeta_{l,tt}D_t(\xi_1) - \zeta_{l,tx}D_t(\xi_2) - \zeta_{l,ty}D_t(\xi_3),$$
$$\eta_l^{xx} = D_x\eta_l^x - \zeta_{l,tx}D_x(\xi_1) - \zeta_{l,xx}D_x(\xi_2) - \zeta_{l,xy}D_x(\xi_3),$$
$$\eta_l^{yy} = D_y\eta_l^y - \zeta_{l,ty}D_y(\xi_1) - \zeta_{l,xy}D_y(\xi_2) - \zeta_{l,yy}D_y(\xi_3),$$
$$l = 1, 2, 3. \tag{11}$$

The expansion of the total derivatives D_t, D_x and D_y is $D_m = \frac{\partial}{\partial m} + \zeta_{l,m}\frac{\partial}{\partial \zeta_l} + \zeta_{l,mm}\frac{\partial}{\partial \zeta_{l,m}} + \cdots,\quad m \in \{t, x, y\}$ where the summation is over repeated indices l. In order to find the infinitesimal coordinates ξ_l, η_l for $l = 1, 2, 3$ of the operator (8) we apply its second extension $\mathbf{X}^{[2]}$ on each equation of system (1). For example, in case of first equation of the above system the invariance criterion reads as

$$\mathbf{X}^{[2]}(u_x + v_y)\ |_{u_x+v_y=0} = 0, \tag{12}$$

where $|_{u_x+v_y=0}$ means the above expression is evaluated at first equation of the system (1). The coefficients of u, v, T and their partial derivatives (different powers of the derivatives), that appear in the resulting equation are equated to zero. This provides a system of linear PDEs involving ξ_l, η_l and their partial derivatives. Solving the resulting system in MAPLE we find $\xi_1 = a_6t + a_1$, $\xi_2 = a_4x + a_5t + a_2$, $\xi_3 = \frac{a_6y}{2}$, $\eta_1 = a_4u - a_6u + a_5$, $\eta_2 = -\frac{a_6v}{2}$, $\eta_3 = a_3T + a_7$ where $a_1, a_2, \ldots, a_7$ are constants and to each there corresponds a Lie point symmetry generator, that are

$$\mathbf{X}_1 = \frac{\partial}{\partial t},\ \ \mathbf{X}_2 = \frac{\partial}{\partial x},\ \ \mathbf{X}_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},\ \ \mathbf{X}_4 = T\frac{\partial}{\partial T},$$
$$\mathbf{X}_5 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\ \ \mathbf{X}_6 = t\frac{\partial}{\partial t} + \frac{y}{2}\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - \frac{v}{2}\frac{\partial}{\partial v},\ \ \mathbf{X}_7 = \frac{\partial}{\partial T}. \tag{13}$$

Lie point symmetries of DEs and their systems can alternatively be obtained using MAPLE package (PDEtools) with the (Infinitesimals command), this package employs the algorithm explained above for obtaining Lie point symmetries. System (1) admits a 7-dimensional Lie point symmetry algebra that is spanned by the above generators. These symmetry generators should also leave all conditions (2) invariant. To investigate this invariance these generators are employed on each component of the conditions through the criterion (12). In these invariance criteria we apply zeroth extension of $\mathbf{X}_i$ for $i = 1, 2, \ldots, 7$ except the cases which involve first order derivatives like u_y and T_y where we apply first extension of the symmetry generators readable from (9) to (10). This exercise reveals certain conditions on the film thickness $h(t)$, velocity of the stretching sheet $U(x, t)$ and temperature $T_s(x, t)$. Moreover it infers that $\mathbf{X}_7$ does not leave all components of conditions (2) invariant. However, when it appears in a linear combination with other generators then becomes compatible with all conditions (2) in accordance with the invariance criterion (12). Therefore, in the remaining work we are considering all $\mathbf{X}_1 - \mathbf{X}_7$ and their linear combinations for fabricating similarity transformations and deriving analytic solutions of the system (1).

The derived symmetries and their linear combinations are used to construct invariants that finally lead to similarity transformations. A zeroth order invariant associated with the Lie algebra $\mathcal{L}^7$ spanned by $\mathbf{X}_i$, $i = 1, 2, \ldots, 7$ is a function J of the independent and dependent variables of the considered system of PDEs (1). These are obtainable from an invariance criterion

$$\mathbf{X}_iJ(t, x, y, u, v, T) = 0,\quad i = 1, 2, \ldots, 7, \tag{14}$$

that yields a linear PDE which on solving provides associated invariants. By repeating this procedure twice and labeling the obtained invariants as the new independent and dependent variables we construct the similarity transformations and use them to deduce corresponding systems of ODEs.

3. Construction of similarity transformations and double reductions

We derive similarity transformations through invariants of the symmetry generators associated with a system of differential equations. All the symmetry generators and their linear combinations dictate specific forms of film thickness $h(t)$, stretching surface velocity $U(x, t)$ and temperature of the elastic sheet $T_s(x, t)$, when we apply them on conditions (2). For example, we consider $\mathbf{X}_5$ here, it leaves all other conditions form invariant except $u = U(x, t)$, $T = T_s(x, t)$, for which the invariance criterion (12) reads as

$$\mathbf{X}_5(u - U(x, t))\ |_{u=U(x,t)} = 0,$$
$$\mathbf{X}_5(T - T_s(x, t))\ |_{T=T_s(x,t)} = 0, \tag{15}$$

that leads to linear PDEs $-x\frac{\partial U}{\partial x} + U(x, t) = 0$, $-x\frac{\partial T_s}{\partial x} = 0$. Solving them we obtain $U(x, t) = xF(t)$, $T_s(x, t) = G(t)$ which in the present case brings (2) to the following form

$$y = 0,\ \ u = xF(t),\ \ v = 0,\ \ T = G(t),$$
$$y = h(t),\ \ u_y = T_y = 0,\ \ v = \frac{dh}{dt}. \tag{16}$$

Notice that in these conditions we have $u_y = 0$ and $T_y = 0$ for which we need first extension $\mathbf{X}_5^{[1]}$ of the considered generator, to apply on them for investigation of invariance. This first extension is obtainable from (9) to (10) that, indeed leaves the mentioned components invariant. In Tables 1 and 2 conditions corresponding to symmetry generators (13) and their linear combinations are listed.

In other words system (1) and conditions (2) remain invariant under the symmetry generators and their linear combinations if $h(t)$, $U(x, t)$ and $T_s(x, t)$ has the specific forms given in these tables. For derivation of the invariants we employ the symmetry generator $\mathbf{X}_5$ in the invariance criterion (14). It extends to a PDE $x\frac{\partial J}{\partial x} + u\frac{\partial J}{\partial u} = 0$ that on solving yields invariants $\{t, y, \frac{u}{x}, v, T\}$. Labeling them as new independent z_1, z_2, and dependent variables F_1, F_2, F_3 reveals the following relations

$$z_1 = t,\ \ z_2 = y,\ \ F_1 = \frac{u}{x},\ \ F_2 = v,\ \ F_3 = T. \tag{17}$$

Table 1
Symmetry generators, invariants and corresponding conditions.

Symmetry and invariants	Conditions
$\mathbf{X}_1$	$y = 0,\ u = U(x),\ v = 0,\ T = T_s(x)$
$x,\ y,\ u,\ v,\ T$	$y = C,\ u_y = v = T_y = 0$
$\mathbf{X}_2$	$y = 0,\ u = U(t),\ v = 0,\ T = T_s(t)$
$t,\ y,\ u,\ v,\ T$	$y = h(t),\ u_y = T_y = 0,\ v = \frac{dh}{dt}$
$\mathbf{X}_3$	$y = 0,\ u = U(t) + \frac{x}{t},\ v = 0,\ T = T_s(t)$
$t,\ y,\ u - \frac{x}{t},\ v,\ T$	$y = h(t),\ u_y = T_y = 0,\ v = \frac{dh}{dt}$
$\mathbf{X}_4$	$y = 0,\ u = U(x, t),\ v = 0,\ T = 0$
$t,\ x,\ y,\ u,\ v$	$y = h(t),\ u_y = T_y = 0,\ v = \frac{dh}{dt}$
$\mathbf{X}_5$	$y = 0,\ u = xU(t),\ v = 0,\ T = T_s(t)$
$t,\ y,\ \frac{u}{x},\ v,\ T$	$y = h(t),\ u_y = T_y = 0,\ v = \frac{dh}{dt}$
$\mathbf{X}_6$	$y = 0,\ u = \frac{U(x)}{t},\ v = 0,\ T = T_s(x)$
$x,\ \frac{y}{\sqrt{t}},\ tu,\ \sqrt{t}v,\ T$	$y = C\sqrt{t},\ u_y = T_y = 0,\ v = \frac{C}{2\sqrt{t}}$
$\mathbf{X}_7$	Does not leave $T = T_s(x, t)$ invariant
$t,\ x,\ y,\ u,\ v$	

Table 2
Symmetry generators, invariants and corresponding conditions.

Symmetry and invariants	Conditions
$\mathbf{X}_3+\mathbf{X}_4$	$y=0,\ u=xU(t),\ v=0,\ T=xT_s(t)$
$t,\ y,\ \frac{u}{x},\ v,\ \frac{T}{x}$	$y=h(t),\ u_y=T_y=0,\ v=\frac{dh}{dt}$
$\mathbf{X}_3+\mathbf{X}_6$	$y=0,\ u=\frac{U(x)}{t},\ v=0,\ T=tT_s(x)$
$x,\ \frac{y}{\sqrt{t}},\ tu,\ \sqrt{t}v,\ \frac{T}{t}$	$y=C\sqrt{t},\ u_y=T_y=0,\ v=\frac{C}{2\sqrt{t}}$
$\mathbf{X}_4+\mathbf{X}_6$	$y=0,\ u=U(\frac{x}{t}),\ v=0,\ T=T_s(\frac{x}{t})$
$\frac{x}{t},\ \frac{y}{\sqrt{t}},\ u,\ \sqrt{t}v,\ T$	$y=C\sqrt{t},\ u_y=T_y=0,\ v=\frac{C}{2\sqrt{t}}$
$\mathbf{X}_4+\mathbf{X}_7$	$y=0,\ u=xU(t),\ v=0,\ T=T_s(t)+\ln(x)$
$t,\ y,\ \frac{u}{x},\ v,\ T-\ln(x)$	$y=h(t),\ u_y=T_y=0,\ v=\frac{dh}{dt}$
$\mathbf{X}_5+\mathbf{X}_6$	$y=0,\ u=1+\frac{U(x-t)}{t},\ v=0,\ T=T_s(x-t)$
$x-t,\ \frac{y}{\sqrt{t}},\ -t+tu,\ \sqrt{t}v,\ T$	$y=C\sqrt{t},\ u_y=T_y=0,\ v=\frac{C}{2\sqrt{t}}$
$\mathbf{X}_6+\mathbf{X}_7$	$y=0,\ u=\frac{U(x)}{t},\ v=0,\ T=T_s(x)+\ln(t)$
$x,\ \frac{y}{\sqrt{t}},\ tu,\ \sqrt{t}v,\ T-\ln(t)$	$y=C\sqrt{t},\ u_y=T_y=0,\ v=\frac{C}{2\sqrt{t}}$

These mappings transform system (1) and its conditions that are now in the format (16), to

$$F_1+F_{2,z_2}=0,\quad F_{1,z_1}+F_2F_{1,z_2}+F_1^2-\nu F_{1,z_2z_2}=0,\quad F_{3,z_1}+F_2F_{3,z_2}-\kappa F_{3,z_2z_2}=0, \tag{18}$$

and

$$z_2=0,\ F_1=F(z_1),F_2=0,F_3=G(z_1),\quad z_2=h(z_1),\ F_{1,z_2}=F_{3,z_2}=0,F_2=h_{z_1}. \tag{19}$$

Further for the double reduction symmetries and invariants of the system (18) are derived. This once reduced system admits a four dimensional symmetry algebra spanned by the generators $\mathbf{Y}_1=\partial_{z_1}$, $\mathbf{Y}_2=\partial_{F_3}$, $\mathbf{Y}_3=F_3\partial_{F_3}$, $\mathbf{Y}_4=z_1\partial_{z_1}+\frac{z_2}{2}\partial_{z_2}-F_1\partial_{F_1}-\frac{F_2}{2}\partial_{F_2}$. The symmetry generators $\mathbf{Y}_1$ and $\mathbf{Y}_4$ from this list are engaged further which convert the conditions (16) to the following forms

$$z_2=0,\ F_1=C_1,\ F_2=0,\ F_3=C_2,\quad z_2=C_3,\ F_{1,z_2}=F_2=F_{3,z_2}=0, \tag{20}$$

$$z_2=0,\ F_1=\frac{C_1}{z_1},\ F_2=0,\ F_3=C_2,\quad z_2=C_3\sqrt{z_1},\ F_{1,z_2}=F_{3,z_2}=0,\ F_2=\frac{C_3}{2\sqrt{z_1}}, \tag{21}$$

when $\mathbf{Y}_1$ and $\mathbf{Y}_4$ are applied on (16) through the invariance criterion. The invariants associated with these symmetries are $\{z_2,F_1,F_2,F_3\}$ and $\{\frac{z_2}{\sqrt{z_1}},z_1F_1,\sqrt{z_1}F_2,F_3\}$, respectively. Considering them as new independent and dependent variables as

$$\chi=z_2,\ g_1=F_1,\ g_2=F_2,\ g_3=F_3, \tag{22}$$

and

$$\chi=\frac{z_2}{\sqrt{z_1}},\ g_1=z_1F_1,\ g_2=\sqrt{z_1}F_2,\ g_3=F_3, \tag{23}$$

we perform the double reductions. The transformation (22) and (23) maps system (18) with the associated conditions (20) and (21) to

$$g_2'+g_1=0,\quad g_2g_1'+g_1^2-\nu g_1''=0,\quad g_2g_3'-\kappa g_3''=0,$$

$$\chi=0,\ g_1=C_1,\ g_2=0,\ g_3=C_2,\quad \chi=C_3,\ g_1'=g_2=g_3'=0, \tag{24}$$

and

$$g_2'+g_1=0,\quad (g_2-\frac{\chi}{2})g_1'+g_1(g_1-1)-\nu g_1''=0,\quad (g_2-\frac{\chi}{2})g_3'-\kappa g_3''=0,$$

$$\chi=0,\ g_1=C_1,\ g_2=0,\ g_3=C_2,\quad \chi=C_3,\ g_1'=g_3'=0,\ g_2=\frac{C_3}{2}, \tag{25}$$

respectively.

In the system (6) and its conditions there exist Pr and S, that arise due to positive constants b and α with time^{-1} dimensions, β and ν which denote dimensionless film thickness and kinematic viscosity of fluid respectively, in the associated transformations (5). Therefore, we introduce all of them in the transformations that are constructed through invariants provided by the second symmetries $\mathbf{Y}_1$ and $\mathbf{Y}_4$. For instance, we consider

$$\beta\sqrt{\frac{\alpha\nu}{b}}\eta=\chi,\quad \frac{-b}{\alpha}f'=g_1,\ \beta\sqrt{\frac{\nu b}{\alpha}}f=g_2,\ \theta=g_3, \tag{26}$$

in the invariants associated with $\mathbf{Y}_1$. Now combining it with the transformations (17) and (22), we finally obtain the following invertible mappings of the independent and dependent variables

$$y=\beta\sqrt{\frac{\alpha\nu}{b}}\eta,\ u=-\frac{b}{\alpha}xf',\ v=\beta\sqrt{\frac{\nu b}{\alpha}}f,\ T=\theta, \tag{27}$$

which directly transform system (1) to the following system of ODEs

$$f'''+\gamma(f'^2-ff'')=0,\quad Pr^{-1}\theta''-\gamma f\theta'=0. \tag{28}$$

In the above system prime denotes derivative with respect to η, $Pr=\frac{\nu}{\kappa}$, is the Prandtl number, $S=\frac{\alpha}{b}$, is measure of unsteadiness that is a dimensionless quantity and $\beta^2=\gamma$, is an unknown constant determined via the series solution procedure applied here, as a part of the problem. The conditions (2) transform to

$$f(0)=0,\ \theta(0)=1,\ f'(0)=1,\quad f(1)=0,\ \theta'(1)=0,\quad f''(1)=0, \tag{29}$$

under (27) and assuming $C_1=-b/\alpha$, $C_2=1$ and $C_3=\beta\sqrt{\alpha\nu/b}$. Similarly, for the second generator $\mathbf{Y}_4$ along with $\mathbf{X}_5$ we construct the following transformations

$$y=\beta\sqrt{\frac{\alpha\nu t}{b}}\eta,\ u=-\frac{bx}{\alpha t}f',\ v=\beta\sqrt{\frac{\nu b}{\alpha t}}f,\ T=\theta. \tag{30}$$

These transformations map the system (1) and associated conditions to the following system of ODEs

$$f'''+\gamma\left(\frac{S\eta}{2}-f\right)f''+\gamma f'(S+f')=0,\quad Pr^{-1}\theta''+\gamma\theta'\left(\frac{S\eta}{2}-f\right)=0, \tag{31}$$

and conditions

$$f(0)=0,\ \theta(0)=1,\ f'(0)=1,\quad f(1)=\frac{S}{2},\ \theta'(1)=0,\quad f''(1)=0. \tag{32}$$

In Table 3 we present similarity transformations and perform double reductions of system (1) and conditions (2) using invariants corresponding to linear combination of symmetries from $\mathbf{X}_1-\mathbf{X}_7$ considering two at a time. There are fifteen such combinations possible but we only proceed with those for which $U(x,t)$ and

Table 3
Double reductions.

Case	Symmetry Generator Similarity Transformation	System of ODEs
1	$\mathbf{X}_3+\mathbf{X}_4$	$f'''+\beta^2(f'^2+Sf'-ff''+\frac{S\eta}{2}f'')=0$
	$y=\beta\sqrt{\frac{\alpha\nu(1+t)}{b}}\eta,\ u=-\frac{bx}{\alpha(1+t)}f'$	$Pr^{-1}\theta''+\beta^2((f'-S)\theta+(\frac{S\eta}{2}-f)\theta')=0$
	$v=\beta\sqrt{\frac{b\nu}{\alpha(1+t)}}f,\ T=x(1+t)\theta$	
2	$\mathbf{X}_3+\mathbf{X}_6$	$f'''+\beta^2(f'^2+Sf'-ff''+\frac{S\eta}{2}f'')=0$
	$y=\beta\sqrt{\frac{\alpha\nu t}{b}}\eta,\ u=-\frac{bx}{\alpha t}f'$	$Pr^{-1}\theta''+\beta^2((f'-S)\theta+(\frac{S\eta}{2}-f)\theta')=0$
	$v=\beta\sqrt{\frac{b\nu}{\alpha t}}f,\ T=xt\theta$	
3	$\mathbf{X}_4+\mathbf{X}_6$	$f'''+\beta^2(f'^2+Sf'-ff''+\frac{S\eta}{2}f'')=0$
	$y=\beta\sqrt{\frac{\alpha\nu t}{b}}\eta,\ u=-\frac{bx}{\alpha t}f'$	$Pr^{-1}\theta''+\beta^2((f'+S)\theta+(\frac{S\eta}{2}-f)\theta')=0$
	$v=\beta\sqrt{\frac{b\nu}{\alpha t}}f,\ T=\frac{x}{t}\theta$	
4	$\mathbf{X}_4+\mathbf{X}_7$	$f'''+\beta^2(f'^2+Sf'-ff''+\frac{S\eta}{2}f'')=0$
	$y=\beta\sqrt{\frac{\alpha\nu t}{b}}\eta,\ u=-\frac{bx}{\alpha t}f'$	$Pr^{-1}\theta''+\beta^2(f'-S+(\frac{S\eta}{2}-f)\theta')=0$
	$v=\beta\sqrt{\frac{b\nu}{\alpha t}}f,\ T=\ln tx+\theta$	
5	$\mathbf{X}_5+\mathbf{X}_6$	$f'''+\beta^2(f'^2+Sf'-ff''+\frac{S\eta}{2}f'')=0$
	$y=\beta\sqrt{\frac{\alpha\nu t}{b}}\eta,\ u=1-\frac{b(x-t)}{\alpha t}f'$	$Pr^{-1}\theta''+\beta^2(f'\theta+(\frac{S\eta}{2}-f)\theta')=0$
	$v=\beta\sqrt{\frac{b\nu}{\alpha t}}f,\ T=(x-t)\theta$	
6	$\mathbf{X}_6+\mathbf{X}_7$	$f'''+\beta^2(f'^2+Sf'-ff''+\frac{S\eta}{2}f'')=0$
	$y=\beta\sqrt{\frac{\alpha\nu t}{b}}\eta,\ u=-\frac{bx}{\alpha t}f'$	$Pr^{-1}\theta''+\beta^2(f'-S+(\frac{S\eta}{2}-f)\theta')=0$
	$v=\beta\sqrt{\frac{b\nu}{\alpha t}}f,\ T=\ln(xt)+\theta$	

$T_s(x,t)$ remain functions of both time and space variables, that clearly is not the case with all the conditions that correspond to single symmetries presented in Table 1 while all in Table 2 fall in this category. Hence we investigate and construct analytic solutions for all cases in Table 2.

4. Analytic solutions using homotopy analysis method (HAM)

In this section analytic solutions for the systems of ODEs derived in the Table 3 are constructed using HAM (see, e.g. [13]). In order to apply HAM on systems of the form (28) and (31) initial functions $f_0(\eta)$ and $\theta_0(\eta)$ are derived using the conditions associated with these systems. Here we have two type of conditions namely (29) and (32), hence we obtain two sets of initial functions that read as

$$f_0(\eta)=\eta-\frac{3}{2}\eta^2+\frac{1}{2}\eta^3,\quad \theta_0(\eta)=1, \tag{33}$$

and

$$f_0(\eta)=\eta+\frac{3S-6}{4}\eta^2+\frac{2-S}{4}\eta^3,\quad \theta_0(\eta)=1. \tag{34}$$

The following zeroth order deformation equations are constructed then, which we present here in terms of system (31)

$$\begin{aligned}(1-p)L_f[F(\eta,p)-f_0(\eta)] &= ph_fH_f(\eta)\Big(F'''+\Gamma\Big(\frac{S\eta}{2}-F\Big)F''\\ &\quad+\Gamma(S+F')F'\Big),\\ (1-p)L_\theta[\Theta(\eta,p)-\theta_0(\eta)] &= ph_\theta H_\theta(\eta)\Big(Pr^{-1}\Theta''\\ &\quad+\Gamma\Big(\frac{S\eta}{2}-F\Big)\Theta'\Big),\end{aligned} \tag{35}$$

where prime denotes differentiation with respect to η, $p\in[0,1]$ is the embedding parameter, h_f, h_θ and $H_f(\eta)$, $H_\theta(\eta)$ are the auxiliary parameters and functions, respectively and $L_f=\frac{\partial^3}{\partial\eta^3}$ and $L_\theta=\frac{\partial^2}{\partial\eta^2}$. The auxiliary parameters are adjusted to make the Maclaurin series for $F(\eta,p)$, $\Theta(\eta,p)$ and $\Gamma(p)$ converge at $p=1$ when expanded with respect to p. The conditions associated with (35) read as

$$\begin{aligned}&F(0,p)=0,\ F'(0,p)=1,\ \Theta(0,p)=1,\\ &F(1,p)=\frac{S}{2},\ F''(1,p)=0,\ \Theta'(1,p)=0.\end{aligned} \tag{36}$$

The following holds for $p=0$ and $p=1$

$$\begin{aligned}&F(\eta,0)=f_0(\eta),\ \Theta(\eta,0)=\theta_0(\eta),\\ &F(\eta,1)=f(\eta),\ \Theta(\eta,1)=\theta(\eta),\ \Gamma(1)=\gamma,\end{aligned} \tag{37}$$

which implies a variation in p (from 0 to 1) varies $F(\eta,p)$ and $\Theta(\eta,p)$ (from $f_0(\eta)$ and $\theta_0(\eta)$) to

$$f(\eta)=f_0(\eta)+\sum_{q=1}^{\infty}f_q(\eta),\quad \theta(\eta)=\theta_0(\eta)+\sum_{q=1}^{\infty}\theta_q(\eta), \tag{38}$$

along with $\Gamma(p)$ that goes from γ_0 to $\gamma=\gamma_0+\sum_{q=1}^{\infty}\gamma_q$ where

$$\begin{aligned}&f_q(\eta)=\frac{1}{q!}\frac{\partial^qF(\eta,p)}{\partial p^q}\Big|_{p=0},\quad \theta_q(\eta)=\frac{1}{q!}\frac{\partial^q\Theta(\eta,p)}{\partial p^q}\Big|_{p=0},\\ &\gamma_q=\frac{1}{q!}\frac{\partial^q\Gamma(p)}{\partial p^q}\Big|_{p=0}.\end{aligned} \tag{39}$$

From the zeroth order deformation equations we obtain q^{th}–order deformation equations of the form

$$\begin{aligned}&L_f[f_q(\eta)-\lambda_qf_{q-1}(\eta)]=h_fH_f(\eta)\zeta_q(\eta),\\ &L_\theta[\theta_q(\eta)-\lambda_q\theta_{q-1}(\eta)]=h_\theta H_\theta(\eta)\vartheta_q(\eta),\end{aligned} \tag{40}$$

by differentiating both equations of (35) q-times with respect to p, dividing it by $q!$ and setting $p=0$. In the above equations λ_q is 0 or 1 if $q=1$ or $q>1$ respectively and

$$\begin{aligned}\zeta_q(\eta)&=f'''_{q-1}-\sum_{n=0}^{q-1}\gamma_{q-1-n}\sum_{j=0}^{n}(f_jf''_{n-j}-f'_jf'_{n-j})+\frac{S\eta}{2}\sum_{n=0}^{q-1}\gamma_nf''_{q-1-n}\\ &\quad+S\sum_{n=0}^{q-1}\gamma_nf'_{q-1-n},\\ \vartheta_q(\eta)&=Pr^{-1}\theta''_{q-1}-\sum_{n=0}^{q-1}\gamma_{q-1-n}\sum_{j=0}^{n}f_j\theta'_{n-j}+\frac{S\eta}{2}\sum_{n=0}^{q-1}\gamma_nf'_{q-1-n}.\end{aligned} \tag{41}$$

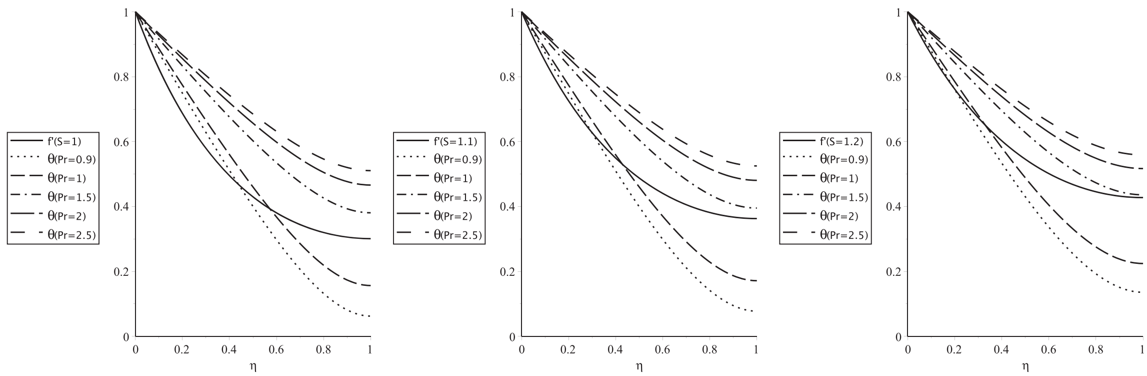

Fig. 1. Case (1) and (2)-plots of $f'(\eta)$ and $\theta(\eta)$ for $h_f = -1$ and $h_\theta = 0.2$.

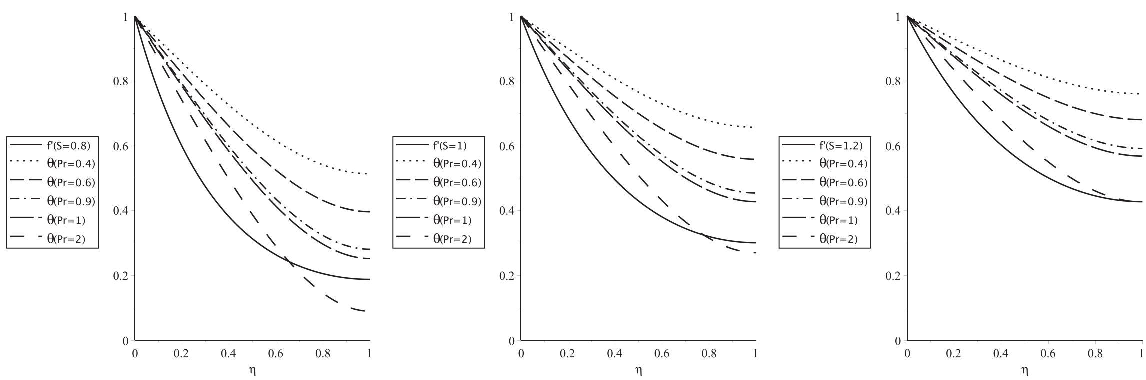

Fig. 2. Case (3)-plots of $f'(\eta)$ and $\theta(\eta)$ for $h_f = -1$ and $h_\theta = -0.2$.

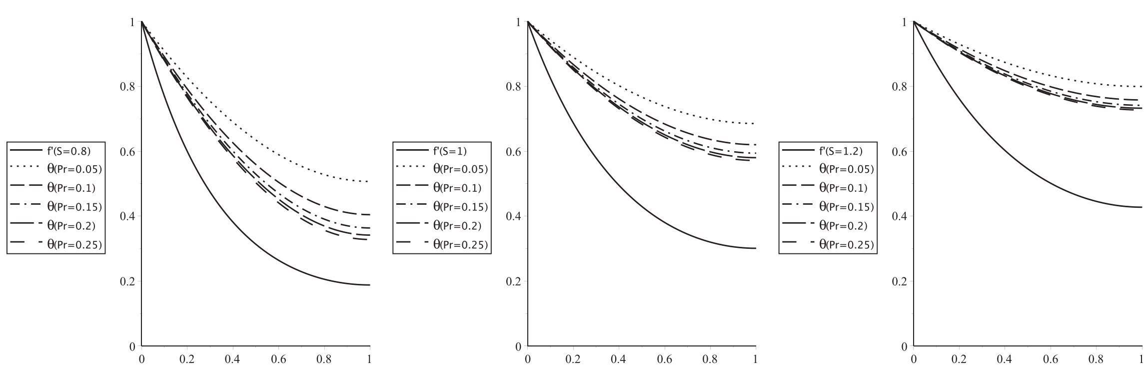

Fig. 3. Case (4)-plots of $f'(\eta)$ and $\theta(\eta)$ for $h_f = -1$ and $h_\theta = -0.007$.

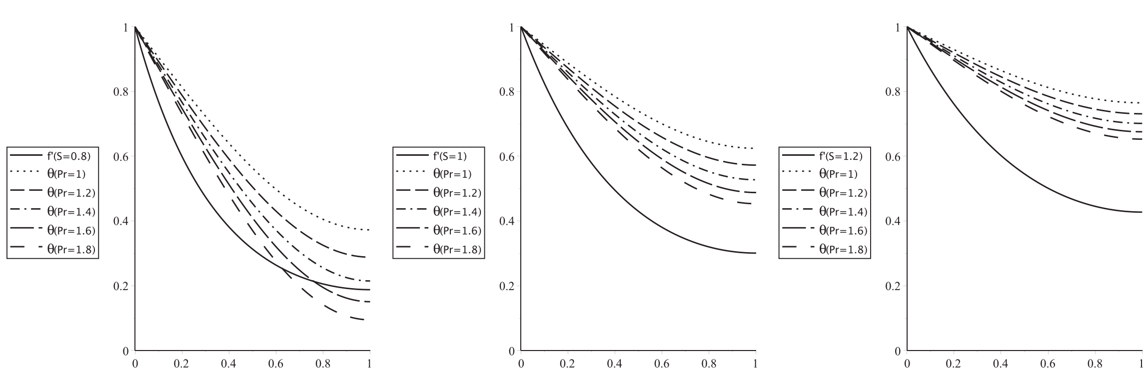

Fig. 4. Case (5)-plots of $f'(\eta)$ and $\theta(\eta)$ for $h_f = -1$ and $h_\theta = -0.3$.

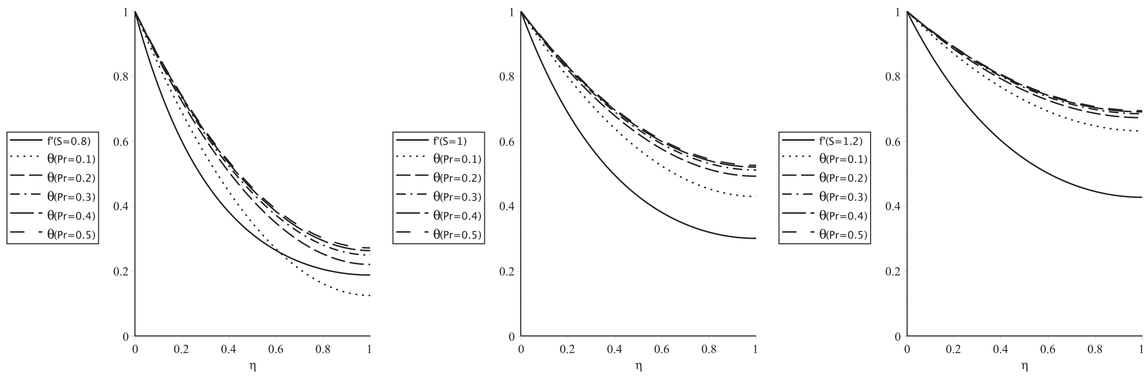

Fig. 5. Case (6)-plots of $f'(\eta)$ and $\theta(\eta)$ for $h_f = -1$ and $h_\theta = 0.008$.

The associated conditions are given by

$$f_q(0) = 0,\ f_q'(0) = 0,\ f_q(1) = 0,\ f_q''(1) = 0,\ \theta_q(0) = 0,$$
$$\theta_q'(1) = 0. \tag{42}$$

Solutions of (40) can be written as

$$f_q(\eta) = \bar{f}_q(\eta) + \lambda_q f_{q-1}(\eta) + C_6 + C_7\eta + C_8\eta^2,$$
$$\theta_q(\eta) = \bar{\theta}_q(\eta) + \lambda_q \theta_{q-1}(\eta) + C_9 + C_{10}\eta, \tag{43}$$

where $\bar{f}_q(\eta)$ and $\bar{\theta}_q(\eta)$ are obtained by integrating (40) and $C_6, \ldots, C_{10}$ are constants of integration. They are determined through the conditions (42) that also provide γ_{q-1}. Solving the qth-order deformation equations successively for $q = 1, 2, 3, \ldots, Q$ yields the Qth order analytic solutions of the form

$$f(\eta) \approx \sum_{q=0}^{Q} f_q(\eta), \quad \theta(\eta) \approx \sum_{q=0}^{Q} \theta_q(\eta), \quad \gamma \approx \sum_{q=0}^{Q} \gamma_q. \tag{44}$$

We use MAPLE for the above computations and construct analytic solutions upto 10th-order of approximation for all cases stated in Table 3. The results are shown in Figs. 1–5.

5. Conclusion

Lie symmetries of the model governing flow and heat transfer in a thin film over an unsteady stretching sheet that is a system of first and second order PDEs with three independent and three dependent variables, are obtained. These symmetry generators constitute a 7-dimensional Lie algebra. By determining invariants for two dimensional sub-algebras of the derived Lie algebra, we constructed similarity transformations. With these transformations, we achieve double reductions of the considered model that transform it into systems of ODEs. For some Lie symmetries and their linear combinations, we found that the velocity and temperature of the stretching sheet are either constants or functions of only t or x. In those cases where we get both U and T_s as functions of both time and space variables, we proceed further. In such cases, we apply HAM on the derived systems of ODEs to find analytic solutions, that are summarized with the help of graphs in Figs. 1–5. There are six such cases that are dealt with HAM to obtain analytic solutions up to tenth order of approximation, by considering the auxiliary functions $H_f(\eta) = H_\theta(\eta) = 1$, different values for the unsteadiness parameter $S(0.8 \to 1.5)$, the Prandtl number $Pr(0.05 \to 3)$ and obtaining values for the auxiliary parameters h_f, h_θ through h-curves [4] that make the solutions converge. For case (1) and (2) velocity and temperature distributions are shown in Fig. 1 to increase with the increment in S and Pr. Figs. 2–4 show velocity and temperature distributions of the cases (3), (4), and (5) which decrease with an increment in the Prandtl number while they increase with S. The solution for the last case is plotted in Fig. 5 and it shows an increment in the velocity and temperature with increment in S and Pr.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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