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Reduction of systems of two nonlinear parabolic type partial differential equations to solvable forms using differential invariants

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ABSTRACT

Differential invariants for linear and nonlinear ordinary and partial differential equations have been derived using Lie infinitesimal method. These invariants help in reducing differential equations to their simplest possible solvable forms through invertible transformations of the dependent and independent variables. Here we derive differential invariants for a class of systems of two second order nonlinear parabolic type partial differential equations. Deduced invariants are shown to reveal solvable forms of these systems of PDEs that are much simpler than the considered general systems of nonlinear parabolic type PDEs.

1. Introduction

Lie infinitesimal method is one of the most efficient methods which can be applied to any class of differential equations [1,2]. This method was originated by Norwegian mathematician Sophus Lie [3] in 1870s. He formulate the theory of continuous groups named as *Lie groups*. Lie group method is extensively used in various fields of mathematics, physics, mechanics and applied sciences to solve linear and nonlinear problems developed in terms of differential equations. The basic aim of Lie group analysis is to get one or several parameters of local continuous group of transformations on the space of dependent and independent variables that leave differential equations invariant and thereafter utilize them to acquire reductions and the so-called similarity solutions through differential invariants [4–7].

Invariants of a differential equation are mathematical expressions written in terms of its coefficients, while differential invariants are those that also involve derivatives of the coefficients. Equivalence transformations play an essential role in finding invariants. A transformation that preserves form of a differential equation is named as an equivalence transformation. The set of all equivalence transformations for a differential equation makes a continuous group. The technique to determine the set of equivalence transformations for a differential equation is known as *Lie infinitesimal method*. Lie introduced the theory of differential invariants and later on Ovsiannikov [8] and Ibragimov [9,10] systematically developed infinitesimal method to calculate invariants of the algebraic and differential equations. Joint invariants are those differential invariants that are derived under transformations of both the dependent and independent variables while those that are deduced under transformation of only the dependent or independent variables are known as semi invariants. Ibragimov derived semi invariants for the linear elliptic, hyperbolic and parabolic PDEs under transformations of only the dependent variable [11].

In this paper we consider a system of two second order nonlinear parabolic type PDEs

$$\begin{aligned} &w_m + a(m,s,w,\upsilon)w_{ss} + b(m,s,w,\upsilon,w_s,\upsilon_s) = 0,\\ &\upsilon_m + c(m,s,w,\upsilon)\upsilon_{ss} + d(m,s,w,\upsilon,w_s,\upsilon_s) = 0, \end{aligned} \quad (1)$$

where m, and s, in subscript denote partial derivatives, e.g, $w_m = \frac{\partial w}{\partial m},\ \upsilon_{ss} = \frac{\partial^2 \upsilon}{\partial s^2}$,. Coefficients a, b, c, d are arbitrary and smooth functions of their arguments. We apply Lie infinitesimal method to construct the set of equivalence transformations for (1), which enables derivation of the associated differential invariants. As an application of the obtained invariants we provide a few examples to demonstrate the claimed reductions using transformations of dependent, independent and only the dependent variables.

The outline of this paper is as follows. Second section is on derivation of the equivalence transformations associated with the considered systems. Subsequent two sections are on deduction of

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the joint and semi differential invariants for systems of parabolic type PDEs. Conclusion is given in the last section.

2. Equivalence transformations

An equivalence transformation of (1) is an invertible mapping of the dependent and independent variables which transforms (1) into itself. Lie infinitesimal method engages the following operator [11]:

$$\boldsymbol{X} = \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + \mu_1\frac{\partial}{\partial a} + \mu_2\frac{\partial}{\partial b} + \mu_3\frac{\partial}{\partial c} + \mu_4\frac{\partial}{\partial d}, \tag{2}$$

to provide the set of equivalence transformations for (1), where $\xi_k = \xi_k(m,\ s,\ w,\ v)$, $\eta_k = \eta_k(m,\ s,\ w,\ v)$, $\mu_l = \mu_l(m,\ s,\ w,\ v,\ w_s,\ v_s,\ a,\ b,\ c,\ d)$ for $k = 1,\ 2,\ l = 1,\ 2,\ 3,\ 4$. For system (1), second order prolongation of the above generator is needed that reads as [11]:

$$\boldsymbol{X}^{[2]} = \boldsymbol{X} + \eta_1^m\frac{\partial}{\partial w_m} + \eta_2^m\frac{\partial}{\partial v_m} + \eta_1^s\frac{\partial}{\partial w_s} + \eta_2^s\frac{\partial}{\partial v_s} + \eta_1^{ss}\frac{\partial}{\partial w_{ss}} + \eta_2^{ss}\frac{\partial}{\partial v_{ss}}, \tag{3}$$

where

$$\eta_1^m = D_m(\eta_1) - w_m D_m(\xi_1) - w_s D_m(\xi_2),$$

$$\eta_2^m = D_m(\eta_2) - v_m D_m(\xi_1) - v_s D_m(\xi_2),$$

$$\eta_1^s = D_s(\eta_1) - w_m D_s(\xi_1) - w_s D_s(\xi_2),$$

$$\eta_2^s = D_s(\eta_2) - v_m D_s(\xi_1) - v_s D_s(\xi_2),$$

$$\eta_1^{ss} = D_s(\eta_1^s) - w_{ms} D_s(\xi_1) - w_{ss} D_s(\xi_2),$$

$$\eta_2^{ss} = D_s(\eta_2^s) - v_{ms} D_s(\xi_1) - v_{ss} D_s(\xi_2),$$

with

$$\begin{aligned} D_m &= \frac{\partial}{\partial m} + w_m\frac{\partial}{\partial w} + v_m\frac{\partial}{\partial v} + w_{mm}\frac{\partial}{\partial w_m} + v_{mm}\frac{\partial}{\partial v_m} + \dots, \\ D_s &= \frac{\partial}{\partial s} + w_s\frac{\partial}{\partial w} + v_s\frac{\partial}{\partial v} + w_{ss}\frac{\partial}{\partial w_s} + v_{ss}\frac{\partial}{\partial v_s} + \dots\ . \end{aligned} \tag{4}$$

Lie invariance condition for system (1) is

$$\begin{aligned} &\boldsymbol{X}^{[2]}(w_m + a w_{ss} + b)|_{(1)} = 0, \\ &\boldsymbol{X}^{[2]}(v_m + c v_{ss} + d)|_{(1)} = 0. \end{aligned} \tag{5}$$

Applying second order prolonged generator (3) on system (1), gives

$$\eta_1^m + a\eta_1^{ss} + \mu_1 w_{ss} + \mu_2|_{(1)} = 0, \tag{6}$$

$$\eta_2^m + c\eta_2^{ss} + \mu_3 v_{ss} + \mu_4|_{(1)} = 0. \tag{7}$$

Inserting $\eta_1^m,\ \eta_2^m,\ \eta_1^{ss},\ \eta_2^{ss}$, from above expressions and $w_m,\ v_m$, from system (1) in (6) and (7) we obtain two equations gives in appendix. After simplification coefficients of w_{ms}, in Appendix A provides

$$\xi_{1,s} + v_s\xi_{1,v} + w_s\xi_{1,w} = 0, \tag{8}$$

which on further comparing coefficients of w_s, and v_s, yields

$$\xi_{1,s} = \xi_{1,w} = \xi_{1,v} = 0. \tag{9}$$

Likewise, coefficients of $w_{ss}v_s$, $w_{ss}w_s$, and v_{ss} in Appendix A and w_{ss} in Appendix B implies

$$\xi_{2,v} = 0,\ \ \xi_{2,w} = 0,\ \ \eta_{1,v} = 0,\ \ \eta_{2,w} = 0, \tag{10}$$

respectively. Equations in (9) and (10) lead to the following infinitesimal coordinates

$$\xi_1 := \xi_1(m),\ \ \xi_2 := \xi_2(m, s),\ \ \eta_1 := \eta_1(m, s, w),\ \ \eta_2 := \eta_2(m, s, v). \tag{11}$$

Using (11) in Appendix A and B, reduces them to

$$\begin{aligned} &a w_s^2\eta_{1,ww} + 2a w_s\eta_{1,sw} + a\eta_{1,ss} - a w_s\xi_{2,ss} - b\eta_{1,w} - w_s\xi_{2,m} + \eta_{1,m} \\ &-2a w_{ss}\xi_{2,s} + a w_{ss}\xi_{1,m} + b\xi_{1,m} + \mu_1 w_{ss} + \mu_2 = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &c v_s^2\eta_{2,vv} + 2c v_s\eta_{2,sv} + c\eta_{2,ss} - c v_s\xi_{2,ss} - d\eta_{2,v} - v_s\xi_{2,m} + \eta_{2,m} \\ &-2c v_{ss}\xi_{2,s} + c v_{ss}\xi_{1,m} + d\xi_{1,m} + \mu_3 v_{ss} + \mu_4 = 0. \end{aligned} \tag{13}$$

Now coefficients of w_{ss} and remaining terms in (12) as well as the coefficients of v_{ss} and other terms in (13) provide

$$\begin{aligned} \mu_1 &= 2a\xi_{2,s} - a\xi_{1,m}, \\ \mu_2 &= a w_s\xi_{2,ss} + b\eta_{1,w} + w_s\xi_{2,m} - a w_s^2\eta_{1,ww} - \eta_{1,m} - 2a w_s\eta_{1,sw} \\ &\quad - a\eta_{1,ss} - b\xi_{1,m}, \\ \mu_3 &= 2c\xi_{2,s} - c\xi_{1,m}, \\ \mu_4 &= c v_s\xi_{2,ss} + d\eta_{2,v} + v_s\xi_{2,m} - c v_s^2\eta_{2,vv} - \eta_{2,m} - 2c v_s\eta_{2,sv} \\ &\quad - c\eta_{2,ss} - d\xi_{1,m}. \end{aligned} \tag{14}$$

All components in (11) and (14) constitute the set of equivalence transformations for (1), that will be engaged in the following section to derive invariants.

3. Joint differential invariants

In this section joint differential invariants for system (1) are derived under transformations of both the dependent and independent variables. Firstly, zeroth order invariants are investigated through the invariance criterion $\boldsymbol{X}J(m, s, w, v, a, b, c, d) = 0$, where $\boldsymbol{X}$, is the operator given in (2) along with the coordinates (11) and (14). The given invariance test leads to an equation in Appendix C and equating coefficients of $\xi_1,\ \xi_2,\ \eta_1,\ \eta_2,\ \eta_{1,m},\ \eta_{2,m}$, in said equation to zero implies

$$J_m = J_s = J_w = J_v = J_b = J_d = 0. \tag{15}$$

Hence, $J = J(a, c)$. Further, the terms with $\xi_{1,m},\ \xi_{2,s}$ provide the following equations

$$-aJ_a - cJ_c = 0,\ \ 2aJ_a + 2cJ_c = 0, \tag{16}$$

after solving above system of equations we get one zeroth order invariant, i.e., $J_1 = \frac{c}{a}$. For derivation of first order differential invariants one has to consider the following invariance criterion

$$\boldsymbol{X}^{[1]}J\big(m, s, w, v, a, b, c, d, a_i,\ b_j,\ c_i,\ d_j\big) = 0,\ \ i \in \{m, s, w, v\},\ \ j \in \{m, s, w, v, w_s, v_s\}. \tag{17}$$

Here $\boldsymbol{X}^{[1]}$, denotes first extension of $\boldsymbol{X}$, that is obtainable from the formula

$$\boldsymbol{X}^{[1]} = \boldsymbol{X} + \mu_{1,i}\frac{\partial}{\partial a_i} + \mu_{2,j}\frac{\partial}{\partial b_j} + \mu_{3,i}\frac{\partial}{\partial c_i} + \mu_{4,j}\frac{\partial}{\partial d_j}. \tag{18}$$

Notice that here the first extension of (2) is different from the one obtained earlier to derive equivalence transformations for (1). There the extensions were with respect to η_k's and here they are according to μ_l's, for $k = 1, 2,$ and $l = 1, 2, 3, 4,$ respectively, to act on partial derivatives of the arbitrary coefficients appearing in system (1). In above equation $\mu_{1,i}$, $\mu_{2,j}$, $\mu_{3,i}$, $\mu_{4,j}$ are expressed as

$$\begin{aligned} \mu_{1,i} &= \tilde{D}_i(\mu_1) - a_m\tilde{D}_i(\xi_1) - a_s\tilde{D}_i(\xi_2) - a_w\tilde{D}_i(\eta_1) - a_v\tilde{D}_i(\eta_2), \\ \mu_{2,j} &= \tilde{D}_j(\mu_2) - b_m\tilde{D}_j(\xi_1) - b_s\tilde{D}_j(\xi_2) - b_w\tilde{D}_j(\eta_1) - b_v\tilde{D}_j(\eta_2) \\ &\quad - b_{w_s}\tilde{D}_j(\eta_1^s) - b_{v_s}\tilde{D}_j(\eta_2^s), \\ \mu_{3,i} &= \tilde{D}_i(\mu_3) - c_m\tilde{D}_i(\xi_1) - c_s\tilde{D}_i(\xi_2) - c_w\tilde{D}_i(\eta_1) - c_v\tilde{D}_i(\eta_2), \\ \mu_{4,j} &= \tilde{D}_j(\mu_4) - d_m\tilde{D}_j(\xi_1) - d_s\tilde{D}_j(\xi_2) - d_w\tilde{D}_j(\eta_1) - d_v\tilde{D}_j(\eta_2) \\ &\quad - d_{w_s}\tilde{D}_j(\eta_1^s) - d_{v_s}\tilde{D}_j(\eta_2^s), \end{aligned} \tag{19}$$

since $i \subset j$, therefore $\tilde{D}_i \subset \tilde{D}_j$ and generally total derivative operator is defined by

$$\tilde{D}_j = \frac{\partial}{\partial j} + a_i\frac{\partial}{\partial a} + a_{i,i}\frac{\partial}{\partial a_i} + \cdots + b_j\frac{\partial}{\partial b} + b_{j,j}\frac{\partial}{\partial b_j} + \ldots + c_i\frac{\partial}{\partial c} + c_{i,i}\frac{\partial}{\partial c_i} + \ldots + d_j\frac{\partial}{\partial d} + d_{j,j}\frac{\partial}{\partial d_j} + \ldots. \tag{20}$$

Upon equating to zero the terms with ξ_1, $\xi_2, \eta_1, \eta_2, \eta_{1,mm}, \eta_{1,sss}, \eta_{1,www}, \eta_{2,mm}, \eta_{2,sss}, \eta_{2,vvv}$, and simplifying we get

$$J_m = J_s = J_w = J_v = J_{b_m} = J_{b_s} = J_{b_w} = J_{d_m} = J_{d_s} = J_{dv} = 0, \tag{21}$$

hence, $J = J(a, b, c, d, a_m, a_s, a_w, a_v, b_v, b_{w_s}, b_{v_s}, c_m, c_s, c_w, c_v, d_w, d_{w_s}, d_{v_s})$. Now the terms with $\xi_{1,m}, \xi_{1,mm}, \xi_{2,m}, \xi_{2,s}, \xi_{2,ms}, \xi_{2,ss}, \eta_{1,m}, \eta_{1,s}, \eta_{1,w}, \eta_{1,ss}, \eta_{1,sw}, \eta_{1,ww}, \eta_{2,m}, \eta_{2,s}, \eta_{2,v}, \eta_{2,ss}, \eta_{2,sv}, \eta_{2,vv}$, leads to a system in Appendix D. After solving said system in MAPLE we get the following first order joint differential invariants along with J_1

$$J_2 = \frac{c_w}{a_w},\ J_3 = \frac{c_v}{a_v},\ J_4 = \frac{b_{v_s}a_w^2}{d_{w_s}a_v^2}. \tag{22}$$

Proposition 1. A system of two second order nonlinear parabolic type PDEs (1) has one zeroth and three first order joint differential invariants (22).

Applications: To illustrate the invariance criteria developed and corresponding reductions of systems of the form (1) we present the following examples.

1 A coupled system of parabolic type PDEs

$$\begin{aligned} &w_m + \left(\frac{sw^2 + sv}{m}\right)w_{ss} + \left(\frac{sw^2 + sv}{mw}\right)w_s^2 + \left(\frac{2w^2 - s + 2v}{m}\right)w_s = 0, \\ &v_m - \left(\frac{s}{m}\right)v_s + \left(\frac{2w}{m}\right)w_s + \frac{w^2}{ms} = 0, \end{aligned} \tag{23}$$

reduces to

$$\begin{aligned} &u_t + (u + c)u_{xx} = 0, \\ &c_t + u_x = 0, \end{aligned} \tag{24}$$

by means of transformations

$$m = t,\ s = \frac{x}{t},\ w = \sqrt{\frac{u}{x}},\ v = \frac{c}{x}. \tag{25}$$

The above mapping is possible because the joint invariants for (23) and (24) are same, i.e.,

$$J_1 = J_2 = J_3 = J_4 = 0. \tag{26}$$

2 Consider a nonlinear system of parabolic type PDEs

$$\begin{aligned} &w_m + \left(\frac{swv\sin(m)}{m^2}\right)w_{ss} - \left(\frac{s}{m}\right)w_s + \left(\frac{s}{m\sin(m)}\right)v_s + w\cot(m) \\ &+\left(\frac{v}{m\sin(m)}\right) = 0, \\ &v_m - \left(\frac{1}{m^2}\right)v_{ss} + \left(\frac{2 - s^2m}{sm^2}\right)v_s + \left(\frac{\sin(m)}{ms}\right)w_s + \frac{v}{m} = 0. \end{aligned} \tag{27}$$

Joint invariants in this case are

$$J_1 = \frac{1}{swv\sin(m)},\ J_2 = 0,\ J_3 = 0,\ J_4 = \frac{s^2v^2}{(\sin m)^2 w^2}. \tag{28}$$

Eq. (27) can be mapped into

$$\begin{aligned} &u_t + ucu_{xx} + c_x = 0, \\ &c_t + c_{xx} + u_x = 0, \end{aligned} \tag{29}$$

under

$$m = t,\ s = \frac{x}{t},\ w = \frac{u}{\sin(t)},\ v = \frac{tc}{x}. \tag{30}$$

Joint invariants for (29) are also same as obtained for (27).

3. Both the following systems

$$\begin{aligned} &w_m + \left(m^2w^2 + \frac{m^2s}{v}\right)w_{ss} + \left(m^2w + \frac{m^2s}{wv}\right)w_s^2 + \left(\frac{s}{m}\right)w_s \\ &-\left(\frac{ms}{2wv^2}\right)v_s + \frac{m}{2wm} = 0, \\ &v_m + \left(\frac{m^2sw^2}{v}\right)v_{ss} - \left(\frac{2m^2sw^2}{v^2}\right)v_s^2 + \left(\frac{sv + 2m^3w^2}{mv}\right)v_s \\ &-\left(\frac{2mwv^2}{s}\right)w_s - \frac{v}{m} = 0, \end{aligned} \tag{31}$$

and

$$\begin{aligned} &u_t + (u + c)u_{xx} + c_x = 0, \\ &c_t + ucc_{xx} + u_x = 0, \end{aligned} \tag{32}$$

are mappable into each other under invertible transformations

$$m = t,\ s = tx,\ w = \sqrt{u},\ v = \frac{tx}{c}. \tag{33}$$

Joint invariants for both of them read as

$$J_1 = \frac{sw^2}{w^2v + s},\ J_2 = \frac{s}{v},\ J_3 = w^2,\ J_4 = 1, \tag{34}$$

$$J_1 = \frac{uc}{u + c},\ J_2 = c,\ J_3 = u,\ J_4 = 1, \tag{35}$$

that are equal under transformation (33).

4. Semi differential invariants

In this section semi invariants associated with (1) are investigated under invertible transformation of only the dependent variable. For the said derivation we consider $\xi_1(m)$, $\xi_2(m, s)$, and all their derivatives equal to zero in the infinitesimal coordinates (11) of (2), i.e., we consider a subgroup of the group of equivalence transformations for (1) that can be written as an operator of the form

$$\boldsymbol{X} = \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + \left(b\eta_{1,w} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss}\right)\frac{\partial}{\partial b}$$

$$+\left(d\eta_{2,v}-cv_s^2\eta_{2,vv}-\eta_{2,m}-2cv_s\eta_{2,sv}-c\eta_{2,ss}\right)\frac{\partial}{\partial d}. \tag{36}$$

For a zeroth order invariant we insert the above generator in $\boldsymbol{X}J(w,v,a,b,c,d)=0$, that gives the following determining equation

$$\eta_1\frac{\partial J}{\partial w}+\eta_2\frac{\partial J}{\partial v}$$
$$+\left(b\eta_{1,w}-aw_s^2\eta_{1,ww}-\eta_{1,m}-2aw_s\eta_{1,sw}-a\eta_{1,ss}\right)\frac{\partial J}{\partial b}$$
$$+\left(d\eta_{2,v}-cv_s^2\eta_{2,vv}-\eta_{2,m}-2cv_s\eta_{2,sv}-c\eta_{2,ss}\right)\frac{\partial J}{\partial d}=0. \tag{37}$$

Equating coefficients of $\eta_1,\ \eta_2,\ \eta_{1,m},\ \eta_{2,m}$ in the above equation to zero provide

$$J_w=J_v=J_b=J_d=0. \tag{38}$$

Solution of the above system provide

$$J_5=a,\ J_6=c. \tag{39}$$

Further, we derive first order invariants by extending the generator (36) according to (18)–(20) up to order one and employing it in the invariance criterion

$$\boldsymbol{X}^{[1]}J\left(w,v,a,b,c,d,a_i,b_j,c_i,d_j\right)=0,\ i\in\{m,s,w,v\},$$
$$j\in\{m,s,w,v,w_s,v_s\}. \tag{40}$$

In the resulting equation we equate the terms with η_1,η_2, and their partial derivatives to zero to get the corresponding system of PDEs. Solving the obtained system in MAPLE we get the following semi differential invariants along with J_5 and J_6

$$J_7=\frac{c_w}{a_w},\ J_8=\frac{c_v}{a_v},\ J_9=\frac{a_vd_{w_s}}{a_w},\ J_{10}=\frac{a_wb_{v_s}}{a_v},$$
$$J_{11}=\frac{1}{cd_{w_s}(a_wc_v-a_vc_w)}\big[-2ac_w^2(a_vd-a_m)+[\{(2dc_v-2c_m)a_w+2a_vcd_w\}a$$
$$-a_vcb_{w_s}d_{w_s}]c_w+2ca_wc_v\left(\tfrac{1}{2}b_{w_s}d_{w_s}-ad_w\right)\big], \tag{41}$$
$$J_{12}=\frac{1}{ab_{v_s}(a_vc_w-a_wc_v)}\big[2ca_v^2(c_wb-c_m)+[\{(-2ba_w+2a_m)c_v-2ac_wb_v\}c$$
$$+ac_wd_{v_s}b_{v_s}]a_v-2aa_wc_v\left(\tfrac{1}{2}b_{v_s}d_{v_s}-cb_v\right)\big],$$

provided that $a_w,a_v,b_{v_s},c_w,c_v,\ d_{w_s}$ is not equal to zero. We thus state the following proposition.

Proposition 2. A system of two second order nonlinear parabolic type PDEs (1) has two zeroth (39) and six first order semi differential invariants (41).

Applications: In this section a few application of the semi invariants are given to illustrate the reductions that can be achieved via derived invariants.

1 The following system of PDEs

$$\begin{aligned}&w_m+\left(\tfrac{w^2s}{v}\right)w_{ss}+\left(\tfrac{ws}{v}\right)w_s^2-\left(\tfrac{s}{2wv^2}\right)v_s+\tfrac{1}{2wv}=0,\\&v_m+\left(\tfrac{w^2v+s}{v}\right)v_{ss}-\left(\tfrac{2w^2v+2s}{v^2}\right)v_s^2+\left(\tfrac{2w^2v+2s}{sv}\right)v_s-\left(\tfrac{2wv^2}{s}\right)w_s=0,\end{aligned} \tag{42}$$

with semi invariants

$$\begin{aligned}&J_5=\tfrac{w^2s}{v},\ J_6=\left(w^2+\tfrac{s}{v}\right),\ J_7=\tfrac{v}{s},\ J_8=\tfrac{1}{w^2},\ J_9=\tfrac{w^2v}{s},\\&J_{10}=\tfrac{s}{w^2v},\ J_{11}=\tfrac{2w^2sw_s}{(w^2v+s)},\ J_{12}=\tfrac{2(w^2v+s)}{s}v_s,\end{aligned} \tag{43}$$

can be mapped to

$$\begin{aligned}&u_t+ucu_{xx}+c_x=0,\\&c_t+(u+c)v_{xx}+u_x=0,\end{aligned} \tag{44}$$

using invertible transformations

$$m=t,\ s=x,\ w=\sqrt{u},\ v=\frac{x}{c}. \tag{45}$$

Semi invariants for (44) are

$$\begin{aligned}&J_5=uc,\ J_6=u+c,\ J_7=\tfrac{1}{c},\ J_8=\tfrac{1}{u},\ J_9=\tfrac{u}{c},\\&J_{10}=\tfrac{c}{u},\ J_{11}=\tfrac{2ucu_x}{(u+c)},\ J_{12}=\tfrac{2(u+c)}{c}c_x,\end{aligned} \tag{46}$$

that are similar to semi invariants (42) under the given point transformations.

2 A coupled system of parabolic type PDEs

$$\begin{aligned}&w_m+\left(\tfrac{1+wv^3}{w}\right)w_{ss}-\left(\tfrac{2+2wv^3}{w^2}\right)w_s^2-\left(3w^2v^2\right)v_s=0,\\&v_m+\left(\tfrac{v^3}{w}\right)v_{ss}+\left(\tfrac{2v^2}{w}\right)v_s^2-\left(\tfrac{1}{3w^2v^2}\right)w_s=0,\end{aligned} \tag{47}$$

can be mapped into

$$\begin{aligned}&u_t+(u+c)u_{xx}+c_x=0,\\&c_t+ucc_{xx}+u_x=0,\end{aligned} \tag{48}$$

using

$$m=t,\ s=x,\ w=\frac{1}{u},\ v=c^{\frac{1}{3}}. \tag{49}$$

The semi invariants for (47) are

$$\begin{aligned}&J_5=\tfrac{1}{w}+v^3,\ J_6=\tfrac{v^3}{w},\ J_7=v^3,\ J_8=\tfrac{1}{w},\\&J_9=J_{10}=1,\ J_{11}=\left(2+2wv^3\right)w_s,\ J_{12}=\tfrac{2v^3}{1+wv^3}v_s,\end{aligned} \tag{50}$$

are same as of (48) under transformations (49).

5. Conclusion

Lie infinitesimal method is employed here to find the set of equivalence transformations and derive corresponding differential invariants for a system of two nonlinear parabolic type PDEs under transformations of both the dependent, independent and only dependent variable. These invariants are shown to reduce the concerned system into its reduced solvable forms via transformations of the dependent, independent and only dependent variables. An enormous reduction in nonlinearities of the considered system is achieved with the derived invariants.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

M. Huzaifa Yaseen: Conceptualization, Data curation. **M. Safdar:** Supervision, Formal analysis. **M. Ijaz Khan:** Writing - original draft, Writing - review & editing. **M.Y. Malik:** Writing - review & editing. **Qiu-Hong Shi:** Visualization.

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Appendix A

$$\eta_{1,m} - w_s\xi_{2,m} - (aw_{ss}+b)\eta_{1,w} + (aw_{ss}+b)\xi_{1,m} - (aw_{ss}+b)(cv_{ss}+d)\xi_{1,v} - (aw_{ss}+b)^2\xi_{1,w} - (cv_{ss}+d)\eta_{1,v} + w_s(cv_{ss}+d)\xi_{2,v}$$
$$+w_s(aw_{ss}+b)\xi_{2,w} + a\big[2w_sv_s\eta_{1,wv} + w_{ss}(aw_{ss}+b)\xi_{1,w} - 2w_s^2v_s\xi_{2,wv} + 2v_s(aw_{ss}+b)\xi_{1,sv} - w_sv_s^2\xi_{2,vv} + 2w_sv_s(aw_{ss}+b)\xi_{1,wv} + 2v_s\eta_{1,sv}$$
$$+2w_s\eta_{1,sw} - w_s\xi_{2,ss} - 2w_s^2\xi_{2,sw} + v_s^2\eta_{1,vv} + v_{ss}\eta_{1,v} + w_s^2\eta_{1,ww} - w_s^3\xi_{2,ww} - 2w_{ms}\xi_{1,s} + w_{ss}\eta_{1,w} - 2w_{ss}\xi_{2,s} + (aw_{ss}+b)\xi_{1,ss} + \eta_{1,ss}$$
$$-w_sv_{ss}\xi_{2,v} - 2w_{ms}v_s\xi_{1,v} - 2w_{ms}w_s\xi_{1,w} - 2w_{ss}v_s\xi_{2,v} - 3w_{ss}w_s\xi_{2,w} + v_s^2(aw_{ss}+b)\xi_{1,vv} + w_s^2(aw_{ss}+b)\xi_{1,ww} + v_{ss}(aw_{ss}+b)\xi_{1,v}$$
$$+2w_s(aw_{ss}+b)\xi_{1,sw} - 2w_sv_s\xi_{2,sv}\big] + \mu_1 w_{ss} + \mu_2 = 0$$

Appendix B

$$\eta_{2,m} - (cv_{ss}+d)\eta_{2,v} - (aw_{ss}+b)\eta_{2,w} + (cv_{ss}+d)\xi_{1,m} + (cv_{ss}+d)^2\xi_{1,v} - (cv_{ss}+d)(aw_{ss}+b)\xi_{1,w} - v_s\xi_{2,m}$$
$$+v_s(cv_{ss}+d)\xi_{2,v} + v_s(aw_{ss}+b)\xi_{2,w} + c$$
$$[2w_sv_s\eta_{2,wv} + 2w_sv_s(cv_{ss}+d)\xi_{1,wv} + 2v_s\eta_{2,sv} + 2w_s\eta_{2,sw} - 2w_sv_s\xi_{2,sw} - 2v_{ms}v_s\xi_{1,v} + 2v_s(cv_{ss}+d)\xi_{1,sv} - 2w_sv_{ss}\xi_{2,w}$$
$$+w_s^2(cv_{ss}+d)\xi_{1,ww} - w_{ss}v_s\xi_{2,w} - 3v_sv_{ss}\xi_{2,v} + 2w_s(cv_{ss}+d)\xi_{1,sw} - 2w_sv_{ms}\xi_{1,w} + w_{ss}(cv_{ss}+d)\xi_{1,w} + v_s^2\eta_{2,vv} + v_s^2(cv_{ss}+d)$$
$$\xi_{1,vv} - 2w_sv_s^2\xi_{2,wv} - v_sw_s^2\xi_{2,ww} - 2v_{ms}\xi_{1,s} + v_{ss}(cv_{ss}+d)\xi_{1,v} + (cv_{ss}+d)\xi_{1,ss} - 2v_{ss}\xi_{2,s} - v_s^3\xi_{2,vv} + v_{ss}\eta_{2,v} - 2v_s^2\xi_{2,sv} + w_s^2\eta_{2,ww}$$
$$+w_{ss}\eta_{2,w} - v_s\xi_{2,ss} + \eta_{2,ss}]$$
$$+\mu_3 v_{ss} + \mu_4 = 0.$$

Appendix C

$$\xi_1\frac{\partial J}{\partial m} + \xi_2\frac{\partial J}{\partial s} + \eta_1\frac{\partial J}{\partial w} + \eta_2\frac{\partial J}{\partial v} + (2a\xi_{2,s} - a\xi_{1,m})\frac{\partial J}{\partial a} + \big(aw_s\xi_{2,ss} + b\eta_{1,w} + w_s\xi_{2,m} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss} - b\xi_{1,m}\big)$$
$$\frac{\partial J}{\partial b} + (2c\xi_{2,s} - c\xi_{1,m})\frac{\partial J}{\partial c} + \big(cv_s\xi_{2,ss} + d\eta_{2,v} + v_s\xi_{2,m} - cv_s^2\eta_{2,vv} - \eta_{2,m} - 2cv_s\eta_{2,sv} - c\eta_{2,ss} - d\xi_{1,m}\big)\frac{\partial J}{\partial d} = 0.$$

Appendix D

$$-b_{v_s}J_{b_{v_s}} - d_{w_s}J_{d_{w_s}} - b_{w_s}J_{b_{w_s}} - aJ_a - bJ_b - cJ_c - dJ_d - 2a_mJ_{a_m} - 2c_mJ_{c_m} - a_sJ_{a_s} - c_sJ_{c_s} - a_wJ_{a_w} - c_wJ_{c_w} - d_wJ_{d_w} - a_vJ_{a_v} - b_vJ_{b_v} - c_vJ_{c_v} - d_{v_s}J_{d_{v_s}} = 0,$$

$$-aJ_{a_m} - cJ_{c_m} = 0,$$

$$J_{b_{w_s}} + w_sJ_b + v_sJ_d - a_sJ_{a_m} - c_sJ_{c_m} + J_{d_{v_s}} = 0,$$

$$b_{v_s}J_{b_{v_s}} + d_{w_s}J_{d_{w_s}} + b_{w_s}J_{b_{w_s}} + 2aJ_a + 2cJ_c + 2a_mJ_{a_m} + 2c_mJ_{c_m} + a_sJ_{a_s} + c_sJ_{c_s} + 2a_wJ_{a_w} + 2c_wJ_{c_w} + 2a_vJ_{a_v} + 2c_vJ_{c_v} + d_{v_s}J_{d_{v_s}} = 0,$$

$$2aJ_{a_m} + 2cJ_{c_m} = 0,$$

$$aJ_{b_{w_s}} + aw_sJ_b + cv_sJ_d + 2aJ_{a_s} + 2cJ_{c_s} + c_wv_sJ_{d_w} + a_vw_sJ_{b_v} + cJ_{d_{v_s}} = 0,$$

$$-J_b - a_wJ_{a_m} - c_wJ_{c_m} = 0,$$

$$a_wJ_{a_s} - c_wJ_{c_s} = 0,$$

$$b_{v_s}J_{b_{v_s}} - d_{w_s}J_{d_{w_s}} + bJ_b - a_wJ_{a_w} - c_wJ_{c_w} - d_wJ_{d_w} + b_vJ_{b_v} = 0,$$

$$-aJ_b - a_vJ_{b_v} = 0,$$

$$-2aJ_{b_{w_s}} - 2aw_sJ_b - d_{w_s}J_{d_w} - 2a_vw_sJ_{b_v} = 0,$$

$$-2aw_sJ_{b_{w_s}} - aw_s^2J_b - d_{w_s}w_sJ_{d_w} - a_vw_s^2J_{b_v} = 0,$$

$$-J_d - a_vJ_{a_m} - c_vJ_{c_m} = 0,$$

$$-a_vJ_{a_s} - c_vJ_{c_s} = 0,$$

$$-b_{v_s}J_{b_{v_s}} + d_{w_s}J_{d_{w_s}} + dJ_d + d_wJ_{d_w} - a_vJ_{a_v} - b_vJ_{b_v} - c_vJ_{c_v} = 0,$$

$$-cJ_d - c_wJ_{d_w} = 0,$$

$$-2cv_sJ_d - 2c_wv_sJ_{d_w} - b_{v_s}J_{b_v} - 2cJ_{d_{v_s}} = 0,$$

$$-cv_s^2J_d - c_wv_s^2J_{d_w} - b_{v_s}v_sJ_{b_v} - 2cv_sJ_{d_{v_s}} = 0.$$

References

[1] Bluman G, Anco S. Symmetry and integration methods for differential equations. Vol. 154. New York: Springer-Verlag Inc; 2002.
[2] Olver P. Applications of lie groups to differential equations, Vol. 107. New York: Springer-Verlag Inc; 1986.
[3] Ibragimov NH, Meleshko SV, Thailert E. Invariants of linear parabolic differential equations. Commun Nonlinear Sci Numer Simul 2008;13.
[4] Stephani H. Differential equations: their solution using symmetries. New york: Cambridge University Press; 1989.
[5] Ibragimov NH. CRC Handbook of lie group analysis of differential equations, Vol. 1, 2 and 3. Boca Raton: CRC Press; 1994–96.
[6] Mahomed FM. Symmetry group classification of ordinary differential equations: survey of some results. Math Methods Appl Sci 2007;30:1995–2012.
[7] Ibragimov NH, Torrisi M, Valenti A. Differential invariants of nonlinear equations vtt=fx,vxvxx+g(x,vx). Commun Nonlinear Sci Numer Simul 2004;9:69–80.
[8] Ovsiannikov LV. Group analysis of differential equations. New York: Academic Press; 1982.
[9] Ibragimov NH. Infinitesimal method in the theory of invariants of algebraic and differential equations. Not South Afr Math Soc 1997;29:61–70.
[10] Ibragimov NH. Elementary Lie group analysis and ordinary differential equations. Chichester: John Wiley and Sons; 1999.
[11] Ibragimov NH. Laplace type invariants for parabolic equations. Nonlinear Dyn 2002;28:125–33.