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Research Article

Solution of Linear and Quadratic Equations Based on Triangular Linear Diophantine Fuzzy Numbers

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This paper is introducing a new concept of triangular linear Diophantine fuzzy numbers (TLDFNs) in a generic way. We first introduce the concept of TLDFNs and then study the arithmetic operations on these numbers. We find a method for the ranking of these TLDFNs. At the end, we formulate the linear and quadratic equations of the types $A + X = B$,$A \cdot X + B = C$, and $A \cdot X^2 + B \cdot X + C = D$ where the elements A, B, C, and D are TLDFNs. We provide a procedure for the solution of these equations using $(\langle \mathrm{s}, \mathrm{t}\rangle, \langle \mathrm{u}, \mathrm{v}\rangle)$-cut and also provide the examples.

1. Introduction

In 1965, Zadeh [1] introduced a new notion of fuzzy set theory. Fuzzy set (FS) theory has been widely acclaimed as offering greater richness in applications than ordinary set theory. Zadeh popularized the concept of fuzzy sets for the first time. There is an area of FS theory, in which the arithmetic operations on FNs play an essential part known as fuzzy equations (FEQs). Fuzzy equations were studied by Sanchez [2], by using extended operations. Accordingly, a profuse number of researchers like Biacino and Lettieri [3], Buckley [4], and Wasowski [5] have studied several approaches to solve FEQs. In [6–9], Buckley and Qu introduced several techniques to evaluate the fuzzy equations of the type $A \cdot X + B = C$ and $A \cdot X^2 + B \cdot X + C = D$, where A, B, C, D, and X are fuzzy numbers (FNs). Jiang [10] studied an approach to solve simultaneous linear equations that coefficients are fuzzy numbers.

Intuitionistic fuzzy sets [11, 12], neutrosophic sets [13, 14], and bipolar fuzzy sets [15] are the generalizations of the fuzzy sets. There are several mathematicians who solved linear and quadratic equations based on intuitionistic fuzzy sets, neutrosophic sets, and bipolar fuzzy sets. Banerjee and Roy [16] studied the intuitionistic fuzzy linear and quadratic equations, Chakraborty et al. [17] studied arithmetic operations on generalized intuitionistic fuzzy number and its applications to transportation problem, Rahaman et al. [18] introduced the solution techniques for linear and quadratic equations with coefficients as Cauchy neutrosophic numbers, and Akram et al. [19–23] introduced some methods for solving the bipolar fuzzy system of linear equations, also see [24–26].

Linear Diophantine fuzzy set [27] is a new generalization of fuzzy set, intuitionistic fuzzy set, neutrosophic set, and bipolar fuzzy set which was introduced by Riaz and Hashmi in 2019. After the introduction of this concept, several mathematicians were attracted towards this concept and worked in this area. Riaz and others studied the decision-making problems related to linear Diophantine fuzzy Einstein aggregation operators [28], spherical linear Diophantine fuzzy sets [29], and linear Diophantine fuzzy relations [30]. Almagrabi et al. [31] introduced a new approach to q-linear Diophantine fuzzy emergency decision support system for COVID-19. Kamac [32] studied linear Diophantine fuzzy algebraic structures.

Motivated by the work of Buckley and Qu [7], we solve the linear and quadratic equations with more generalized fuzzy numbers. As the linear Diophantine fuzzy set, [27] is the more generalized form of fuzzy sets so we studied the linear and quadratic equations based on linear Diophantine fuzzy numbers. In linear Diophantine fuzzy sets, we use the reference parameters, which allow us to choose the grades without any limitation; this helps us in obtaining better results.

In Section 2, we provided the fundamental definitions related to fuzzy sets and linear Diophantine fuzzy sets. In Section 3, we define linear Diophantine fuzzy numbers, in particular, triangular linear Diophantine fuzzy number. Also defined some basic operations on LDF numbers. In Section 4, we provide the ranking of LDF numbers, and in Section 5, we solved linear and quadratic equations based on LDF numbers.

2. Preliminaries and Basic Definitions

This section is devoted to review some indispensable concepts, which are very beneficial to develop the understanding of the prevalent model.

Definition 1 (see [1]). *Let X be a classical set, $\mu_{\mathfrak{M}}$ be a function from X to $[0, 1]$. The MF (membership function) $\mu_{\mathfrak{M}}(\vartheta)$ of a FS (fuzzy set) $\mathfrak{M}$ is defined by*

$$\mathfrak{M} = \{(\vartheta, \mu_{\mathfrak{M}}(\vartheta)) \mid \vartheta \in X \text{ and } \mu_{\mathfrak{M}}(\vartheta) \in 0, 1]\}. \tag{1}$$

Definition 2 (see [33]). *Let $\mathfrak{M}$ be a fuzzy subset of universal set X . Then, $\mathfrak{M}$ is called convex FS if $\forall r, s \in X$ and $\lambda \in [0, 1]$ we have*

$$\mu_{\mathfrak{M}}(\lambda r + (1 - \lambda)s) \geq \min \{\mu_{\mathfrak{M}}(r), \mu_{\mathfrak{M}}(s)\}. \tag{2}$$

Definition 3 (see [1]). *A fuzzy set $\mathfrak{M}$ is said to be normalized if $h(\mathfrak{M}) = 1$.*

Definition 4. An α -level set of a FS $\mathfrak{M}$ is defined as

$$\mathfrak{M}^{\alpha} = \{\vartheta \in X : \mu_{\mathfrak{M}}(\vartheta) \geq \alpha\} \text{ for each } \alpha \in (0, 1]. \tag{3}$$

Definition 5 (see [33]). *A fuzzy subset $\mathfrak{M}$ defined on a set $\mathbb{R}$ (of real numbers) is said to be a FN (fuzzy number) if $\mathfrak{M}$ satisfies the following axioms:*

(a) $\mathfrak{M}$ is continuous: $\mu_{\mathfrak{M}}(t)$ is a continuous function from $\mathbb{R} \longrightarrow [0, 1]$

(b) $\mathfrak{M}$ is normalized: there exists $t \in \mathbb{R}$ such that $\mu_{\mathfrak{M}}(t) = 1$

(c) Convexity of $\mathfrak{M}$: i.e., $\forall t, u, w \in \mathbb{R}$, if $t \leq u \leq w$, then $\mu_{\mathfrak{M}}(u) \geq \min \{\mu_{\mathfrak{M}}(t), \mu_{\mathfrak{M}}(w)\}$

(d) Boundness of support: i.e., $\exists S \in \mathbb{R}$ and $\forall t \in \mathbb{R}$, if $|t| \geq S$, then $\mu_{\mathfrak{M}}(t) = 0$

We denote the set of all FNs by $F_{ns}(\mathbb{R})$.

Now, we study the idea of LDFSs (linear Diophantine fuzzy sets) and their fundamental operations.

Definition 6 (see [27]). *Let X be the universe. A LDFS $£_{\mathfrak{R}}$ on X is defined as follows:*

$$£_{\mathfrak{R}} = \{(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\} \tag{4}$$

where $\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta), \alpha(\vartheta), \beta(\vartheta) \in [0, 1]$ such that

$$\begin{aligned} &0 \leq \alpha(\vartheta)\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \leq 1, \quad \forall \vartheta \in X, \\ &0 \leq \alpha(\vartheta) + \beta(\vartheta) \leq 1. \end{aligned} \tag{5}$$

The hesitation part can be written as

$$\xi\pi_{\mathfrak{R}} = 1 - (\alpha(\vartheta)\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) + \beta(\vartheta)\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)), \tag{6}$$

where ξ is the reference parameter.

We write in short $£_{\mathfrak{R}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle)$ or $£_{\mathfrak{R}} = \langle\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle\rangle$ for

$$£_{\mathfrak{R}} = \{(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\}. \tag{7}$$

Definition 7 (see [27]). *An absolute LDFS on X can be written as*

$$^{1}£_{\mathfrak{R}} = \{(\vartheta, \langle 1, 0\rangle, \langle 1, 0\rangle) : \vartheta \in X\}, \tag{8}$$

and empty or null LDFS can be expressed as

$$^{0}£_{\mathfrak{R}} = \{(\vartheta, \langle 0, 1\rangle, \langle 0, 1\rangle) : \vartheta \in X\}. \tag{9}$$

Definition 8 (see [27]). *Let $£_{\mathfrak{R}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle)$ and $£_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{P}}\rangle, \langle \gamma, \delta\rangle)$ be two LDFSs on the reference set X and $\vartheta \in X$. Then,*

(i) $£^{c}_{\mathfrak{R}} = (\langle \mathfrak{N}^{\nu}_{\mathfrak{R}}, \mathfrak{M}^{\tau}_{\mathfrak{R}}\rangle, \langle \beta, \alpha\rangle)$

(ii) $£_{\mathfrak{R}} = £_{\mathfrak{P}} \Leftrightarrow \mathfrak{M}^{\tau}_{\mathfrak{R}} = \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}} = \mathfrak{N}^{\nu}_{\mathfrak{P}}, \alpha = \gamma, \beta = \delta$

(iii) $£_{\mathfrak{R}} \subseteq £_{\mathfrak{P}} \Leftrightarrow \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \leq \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \geq \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta)$, $\alpha(\vartheta) \leq \gamma(\vartheta), \beta(\vartheta) \geq \delta(\vartheta)$

(iv) $£_{\mathfrak{R}} \cup £_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}\cup\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}\cap\mathfrak{P}}\rangle, \langle \alpha \vee \gamma, \beta \wedge \delta\rangle)$

(v) $£_{\mathfrak{R}} \cap £_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}\cap\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{R}\cup\mathfrak{P}}\rangle, \langle \alpha \wedge \gamma, \beta \vee \delta\rangle)$

where

$$\begin{aligned} &\mathfrak{M}^{\tau}_{\mathfrak{R}\cup\mathfrak{P}}(\vartheta) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \vee \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \mathfrak{M}^{\tau}_{\mathfrak{R}\cap\mathfrak{P}}(\vartheta) = \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \wedge \mathfrak{M}^{\tau}_{\mathfrak{P}}(\vartheta), \\ &\mathfrak{N}^{\nu}_{\mathfrak{R}\cap\mathfrak{P}}(\vartheta) = \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \wedge \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}\cup\mathfrak{P}}(\vartheta) = \mathfrak{N}_{\mathfrak{R}}(\vartheta) \vee \mathfrak{N}^{\nu}_{\mathfrak{P}}(\vartheta). \end{aligned} \tag{10}$$

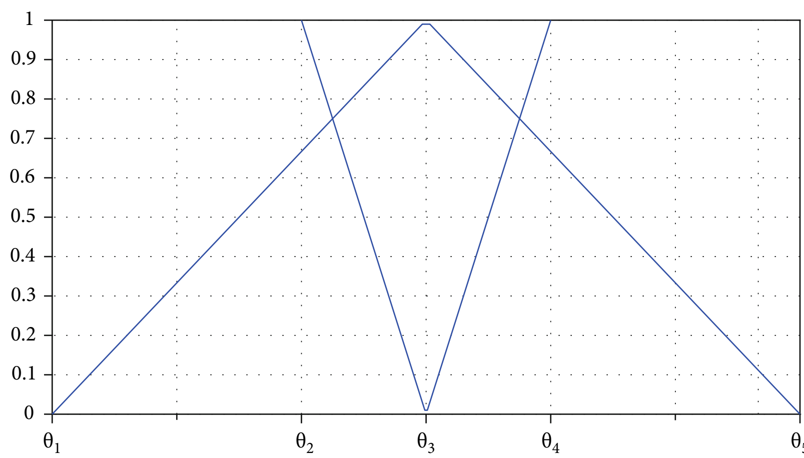

FIGURE 1: The figure of $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$.

Definition 9 (see [27]). *Let* $\pounds_{\mathfrak{R}} = \{(\vartheta, \langle \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta)\rangle, \langle \alpha(\vartheta), \beta(\vartheta)\rangle) : \vartheta \in X\}$ *be an LDFS. For any constants s,t,u ,v* $\in [0, 1]$ *such that* $0 \le su + tv \le 1$ *with* $0 \le u + v \le 1$, *define the* $(\langle s, t\rangle, \langle u, v\rangle)$ *-cut of* $\pounds_{\mathfrak{R}}$ *as follows:*

$$(\pounds_{\mathfrak{R}})^{\langle s,t\rangle}_{\langle u,v\rangle} = \{\vartheta \in X : \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \ge \mathrm{s}, \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \le \mathrm{t}, \alpha(\vartheta) \ge u, \beta(\vartheta)\rangle \le \mathrm{v}\}. \tag{11}$$

3. Triangular LDF Numbers

Here, in this section, we provide definitions and arithmetic operations on LDF numbers (LDFNs).

Definition 10. A LDF number $\pounds_{\mathfrak{R}}$ *is*

(i) a LDF fuzzy subset of the real line $\mathbb{R}$

(ii) normal, i.e., there is any $\vartheta_0 \in \mathbb{R}$ such that $\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta_0) = 1, \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta_0) = 0, \alpha(\vartheta_0) = 1, \beta(\vartheta_0) = 0$

(iii) convex for the membership functions $\mathfrak{M}^{\tau}_{\mathfrak{R}}$ and α, i.e.,

$$\begin{aligned} &\mathfrak{M}^{\tau}_{\mathfrak{R}}(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \ge \min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta_1), \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in [0,1], \\ &\alpha(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \ge \min\{\alpha(\vartheta_1), \alpha(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in [0,1] \end{aligned} \tag{12}$$

(iv) concave for the nonmembership functions $\mathfrak{N}^{\nu}_{\mathfrak{R}}$ and β, i.e.,

$$\begin{aligned} &\mathfrak{N}^{\nu}_{\mathfrak{R}}(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \le \max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta_1), \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in [0,1], \\ &\beta(\lambda\vartheta_1 + (1-\lambda)\vartheta_2) \le \max\{\beta(\vartheta_1), \beta(\vartheta_2)\} \quad \forall \vartheta_1, \vartheta_2 \in \mathbb{R}, \lambda \in 0,1]. \end{aligned} \tag{13}$$

We now provide the 4 types of triangular LDF numbers.

Definition 11. Let $\pounds_{\mathfrak{R}}$ *be a LDFS on* $\mathbb{R}$ *with the following membership functions* ($\mathfrak{M}^{\tau}_{\mathfrak{R}}$ *and* α) *and nonmembership functions* ($\mathfrak{N}^{\nu}_{\mathfrak{R}}$ *and* β)

$$\begin{aligned} \mathfrak{M}^{\tau}_{\mathfrak{R}}(x) &= \begin{cases} \dfrac{x-\vartheta_1}{\vartheta_3-\vartheta_1}, & \vartheta_1 \le x \le \vartheta_3, \\ \dfrac{\vartheta_5 - x}{\vartheta_5-\vartheta_3}, & \vartheta_3 \le x \le \vartheta_5, \\ 0, & \text{otherwise}, \end{cases} \\ \mathfrak{N}^{\nu}_{\mathfrak{R}}(x) &= \begin{cases} \dfrac{\vartheta_3-x}{\vartheta_3-\vartheta_2}, & \vartheta_2 \le x \le \vartheta_3, \\ \dfrac{x - \vartheta_3}{\vartheta_4-\vartheta_3}, & \vartheta_3 \le x \le \vartheta_4, \\ 0, & \text{otherwise}, \end{cases} \\ \alpha(x) &= \begin{cases} \dfrac{x-\vartheta'_2}{\vartheta'_3-\vartheta'_2}, & \vartheta'_2 \le x \le \vartheta'_3, \\ \dfrac{\vartheta'_4 - x}{\vartheta'_4-\vartheta'_3}, & \vartheta'_3 \le x \le \vartheta'_4, \\ 0, & \text{otherwise}, \end{cases} \\ \beta(x) &= \begin{cases} \dfrac{\vartheta'_3-x}{\vartheta'_3-\vartheta'_1}, & \vartheta'_1 \le x \le \vartheta'_3, \\ \dfrac{x - \vartheta'_3}{\vartheta'_5-\vartheta'_3}, & \vartheta'_3 \le x \le \vartheta'_5, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{14}$$

where $\vartheta'_1 \le \vartheta'_2 \le \vartheta'_3 \le \vartheta'_4 \le \vartheta'_5$ for all $x \in \mathbb{R}$. Then, $\pounds_{\mathfrak{R}}$ is called

(i) a triangular LDFN of type-1 if $\vartheta_3 = \vartheta'_3$ and $\vartheta_1 \le \vartheta_2 \le \vartheta_3 \le \vartheta_4 \le \vartheta_5$

(ii) a triangular LDFN of type-2 if $\vartheta_3 \ne \vartheta'_{3'}$ and $\vartheta_1 \le \vartheta_2 \le \vartheta_3 \le \vartheta_4 \le \vartheta_5$

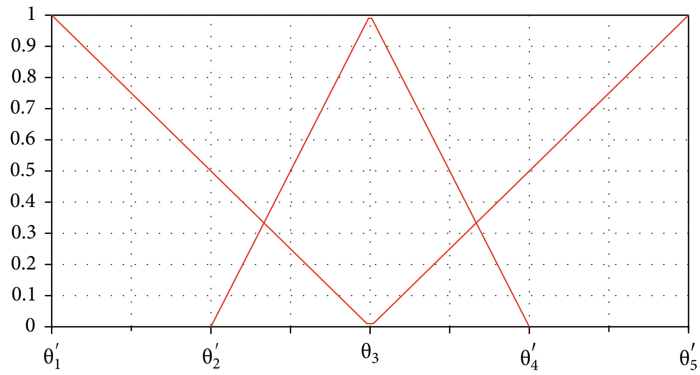

Figure 2: The figure of $(\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5')$.

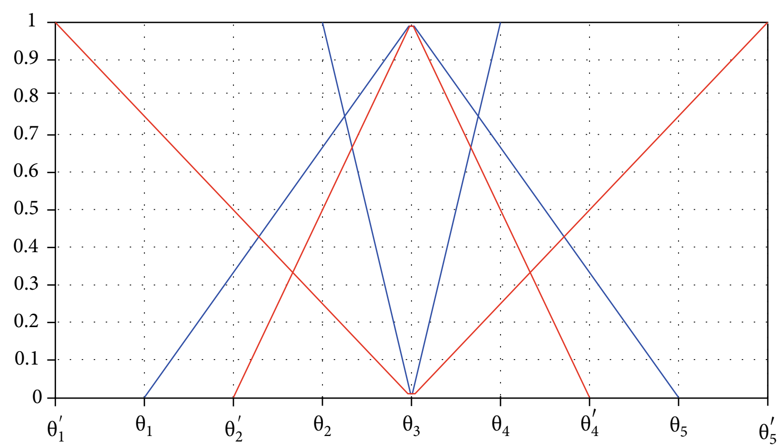

Figure 3: The figure of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$.

(iii) a triangular LDFN of type-3 if $\vartheta_3 = \vartheta_3'$ and $\vartheta_2 \leq \vartheta_1 \leq \vartheta_3 \leq \vartheta_5 \leq \vartheta_4$

(iv) a triangular LDFN of type-4 if $\vartheta_3 \neq \vartheta_3'$ and $\vartheta_2 \leq \vartheta_1 \leq \vartheta_3 \leq \vartheta_5 \leq \vartheta_4$

Throughout the paper, we consider only triangular LDFN of type-1 and we write this type as triangular LDFN (TLDFN). This TLDFN is denoted by

$$\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5), \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5'). \end{cases} \tag{15}$$

The figure of $(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$ is shown in Figure 1.
The figure of $(\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5')$ is shown in Figure 2.
The figure of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is shown in Figure 3.

Remark 12. If we take $\vartheta_1' = \vartheta_2' = \vartheta_1 = \vartheta_2$ and $\vartheta_4' = \vartheta_5' = \vartheta_1 = \vartheta_2$, then both type-1 and type-3 become the same.

Definition 13. Consider a TLDFN $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{smallmatrix} \right.$. Then,

(i) s-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\begin{aligned} \pounds^s_{\mathfrak{R}_{\text{TLDFN}}} &= \{x \in X : \mathfrak{M}^{\tau}_{\mathfrak{R}}(x) \geq s\} = \left[\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)}, \mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s)\right] \\ &= [\vartheta_1 + s(\vartheta_3 - \vartheta_1), \vartheta_5 - s(\vartheta_5 - \vartheta_3)], \end{aligned} \tag{16}$$

(ii) t-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\begin{aligned} \pounds^t_{\mathfrak{R}_{\text{TLDFN}}} &= \{x \in X : \mathfrak{N}^{\nu}_{\mathfrak{R}}(x) \leq t\} = \left[\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)}, \mathfrak{N}^{\bar{\nu}}_{\mathfrak{R}}(t)\right] \\ &= [\vartheta_3 - t(\vartheta_3 - \vartheta_2), \vartheta_3 + t(\vartheta_4 - \vartheta_3)], \end{aligned} \tag{17}$$

(iii) u-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\pounds^{u}_{\mathfrak{R}_{\text{TLDFN}}} = \{x \in X : \alpha(x) \geq u\} = \left[\underline{\alpha(u)}, \bar{\alpha(u)}\right] = \left[\vartheta_2' + u\left(\vartheta_3 - \vartheta_2'\right), \vartheta_4' - u\left(\vartheta_4' - \vartheta_3\right)\right], \tag{18}$$

(iv) ν-cut set of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is a crisp subset of $\mathbb{R}$, which is defined as follows:

$$\pounds^{\nu}_{\mathfrak{R}_{\text{TLDFN}}} = \{x \in X : \beta(x) \leq \nu\} = \left[\underline{\beta(\nu)}, \bar{\beta(\nu)}\right] = \left[\vartheta_3 - \text{v}\left(\vartheta_3 - \vartheta_1'\right), \vartheta_3 + \nu\left(\vartheta_5' - \vartheta_3\right)\right]. \tag{19}$$

We can denote the $(\langle s,t\rangle, \langle u,\nu\rangle)$-cut of $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ by

$$\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right)^{\langle s,t\rangle}_{\langle u,\nu\rangle} = \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)}, \bar{\mathfrak{M}^{\tau}_{\mathfrak{R}}}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)}, \bar{\mathfrak{N}^{\nu}_{\mathfrak{R}}}(t)\right]\right), \\ \left(\left[\underline{\alpha(u)}, \bar{\alpha(u)}\right], \left[\underline{\beta(\nu)}, \bar{\beta(\nu)}\right]\right). \end{cases} \tag{20}$$

We denote the set of all TLDFN on $\mathbb{R}$ by $\pounds_{\mathfrak{R}_{\text{TLDFN}}}(\mathbb{R})$. The arithmetic operations based on extension principle are defined as follows.

Definition 14. Let $\pounds_{\mathfrak{R}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{R}}, \mathfrak{N}^{\nu}_{\mathfrak{R}}\rangle, \langle \alpha, \beta\rangle)$ *and* $\pounds_{\mathfrak{P}} = (\langle \mathfrak{M}^{\tau}_{\mathfrak{P}}, \mathfrak{N}^{\nu}_{\mathfrak{P}}\rangle, \langle \gamma, \delta\rangle)$ *be two TLDFN on* $\mathbb{R}$ *. Then,*

(i) $\pounds_{\mathfrak{R}} + \pounds_{\mathfrak{P}} = \{ \sup_{t=x+y} \{\min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x+y} \{\max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(x), \mathfrak{N}^{\nu}_{\mathfrak{P}}(y)\}\} \sup_{t=x+y} \{\min\{\alpha(x), \gamma(y)\}\} \inf_{t=x+y} \{\max\{\beta(x), \delta(y)\}\}$

(ii) $\pounds_{\mathfrak{R}} - \pounds_{\mathfrak{P}} = \{ \sup_{t=x-y} \{\min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x-y} \{\max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(x), \mathfrak{N}^{\nu}_{\mathfrak{P}}(y)\}\} \sup_{t=x-y} \{\min\{\alpha(x), \gamma(y)\}\} \inf_{t=x-y} \{\max\{\beta(x), \delta(y)\}\}$

(iii) $\pounds_{\mathfrak{R}} \times \pounds_{\mathfrak{P}} = \{ \sup_{t=x\times y} \{\min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x\times y} \{\max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(x), \mathfrak{N}^{\nu}_{\mathfrak{P}}(y)\}\} \sup_{t=x\times y} \{\min\{\alpha(x), \gamma(y)\}\} \inf_{t=x\times y} \{\max\{\beta(x), \delta(y)\}\}$

(iv) $\pounds_{\mathfrak{R}} \div \pounds_{\mathfrak{P}} = \{ \sup_{t=x\div y} \{\min\{\mathfrak{M}^{\tau}_{\mathfrak{R}}(x), \mathfrak{M}^{\tau}_{\mathfrak{P}}(y)\}\} \inf_{t=x\div y} \{\max\{\mathfrak{N}^{\nu}_{\mathfrak{R}}(x), \mathfrak{N}^{\nu}_{\mathfrak{P}}(y)\}\} \sup_{t=x\div y} \{\min\{\alpha(x), \gamma(y)\}\} \inf_{t=x\div y} \{\max\{\beta(x), \delta(y)\}\}$

Definition 15. A TLDFN $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *is said to be positive if and only if* $\vartheta_1 \geq 0$ *and* $\vartheta_1' \geq 0$.

Definition 16. Two TLDFNs $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ *and* $\xi_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta_1', \delta_2', \delta_3, \delta_4', \delta_5') \end{cases}$ *are said to be equal if and only if* $\vartheta_1 = \delta_1, \vartheta_2 = \delta_2, \vartheta_3 = \delta_3, \vartheta_4 = \delta_4, \vartheta_5 = \delta_5, \vartheta_1' = \delta_1', \vartheta_2' = \delta_2', \vartheta_4' = \delta_4'$, *and* $\vartheta_5' = \delta_5'$.

We now define the arithmetic operations on TLDFNs using the concept of interval arithmetic.

Definition 17. Consider two positive TLDFNs $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5) \\ (\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5') \end{smallmatrix} \right.$ *and* $\xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\delta_1,\delta_2,\delta_3,\delta_4,\delta_5) \\ (\delta_1',\delta_2',\delta_3,\delta_4',\delta_5') \end{smallmatrix} \right.$, *then,*

(i) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} + \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1+\delta_1,\vartheta_2+\delta_2,\vartheta_3+\delta_3,\vartheta_4+\delta_4,\vartheta_5+\delta_5) \\ (\vartheta_1'+\delta_1',\vartheta_2'+\delta_2',\vartheta_3+\delta_3,\vartheta_4'+\delta_4',\vartheta_5'+\delta_5') \end{smallmatrix} \right.$

(ii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} - \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1-\delta_5,\vartheta_2-\delta_4,\vartheta_3-\delta_3,\vartheta_4-\delta_2,\vartheta_5-\delta_1) \\ (\vartheta_1'-\delta_5',\vartheta_2'-\delta_4',\vartheta_3-\delta_3,\vartheta_4'-\delta_2',\vartheta_5'-\delta_1') \end{smallmatrix} \right.$

(iii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \times \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} (\vartheta_1\delta_1,\vartheta_2\delta_2,\vartheta_3\delta_3,\vartheta_4\delta_4,\vartheta_5\delta_5) \\ (\vartheta_1'\delta_1',\vartheta_2'\delta_2',\vartheta_3\delta_3,\vartheta_4'\delta_4',\vartheta_5'\delta_5') \end{smallmatrix} \right.$

(iv) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \div \xi_{\mathfrak{R}_{\text{TLDFN}}} = \left\{ \begin{smallmatrix} ((\vartheta_1/\delta_5),(\vartheta_2/\delta_4),(\vartheta_3/\delta_3),(\vartheta_4/\delta_2),(\vartheta_5/\delta_1)) \\ ((\vartheta_1'/\delta_5'),(\vartheta_2'/\delta_4'),(\vartheta_3/\delta_3),(\vartheta_4'/\delta_2'),(\vartheta_5'/\delta_1')) \end{smallmatrix} \right.$

(v) $k \times \pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} \left\{ \begin{smallmatrix} (k\vartheta_1,k\vartheta_2,k\vartheta_3,k\vartheta_4,k\vartheta_5) \\ (k\vartheta_1',k\vartheta_2',k\vartheta_3,k\vartheta_4',k\vartheta_5') \end{smallmatrix} \right. & \text{if } k > 0 \\ \left\{ \begin{smallmatrix} (k\vartheta_5,k\vartheta_4,k\vartheta_3,k\vartheta_2,k\vartheta_1) \\ (k\vartheta_5',k\vartheta_4',k\vartheta_3,k\vartheta_2',k\vartheta_1') \end{smallmatrix} \right. & \text{if } k < 0 \end{cases}$

4. Ranking Function of TLDFNs

There are many methods for defuzzification such as the centroid method, mean of interval method, and removal area method. In this paper, we have used the concept of the mean of interval method to find the value of the membership and nonmembership function of TLDFN.

Consider a TLDFN

$$\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5), \\ (\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5'). \end{cases} \tag{21}$$

The $(\langle s,t\rangle, \langle u,\nu\rangle)$-cut of $\pounds_{\mathfrak{R}_{\text{TLDFN}}}$ is

$$(\pounds_{\mathfrak{R}})^{\langle s,t\rangle}_{\langle u,\nu\rangle} = \{\vartheta \in X : \mathfrak{M}^{\tau}_{\mathfrak{R}}(\vartheta) \geq s, \mathfrak{N}^{\nu}_{\mathfrak{R}}(\vartheta) \leq t, \alpha(\vartheta) \geq u, \beta(\vartheta)\rangle \leq \nu\}, \tag{22}$$

where

$$
\begin{aligned}
\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)} &= \vartheta_1 + s(\vartheta_3 - \vartheta_1),\\
\mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s) &= \vartheta_5 - s(\vartheta_5 - \vartheta_3),\\
\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)} &= \vartheta_3 - t(\vartheta_3 - \vartheta_2),\\
\mathfrak{N}^{\bar{\nu}}_{\mathfrak{R}}(t) &= \vartheta_3 + t(\vartheta_4 - \vartheta_3),\\
\underline{\alpha(u)} &= \vartheta_2' + u\left(\vartheta_3 - \vartheta_2'\right),\\
\bar{\alpha}(u) &= \vartheta_4' - u\left(\vartheta_4' - \vartheta_3\right),\\
\underline{\beta(v)} &= \vartheta_3 - v\left(\vartheta_3 - \vartheta_1'\right),\\
\bar{\beta}(v) &= \vartheta_3 + v\left(\vartheta_5' - \vartheta_3\right).
\end{aligned}
\tag{23}
$$

Now, by the mean of $(\langle s, t\rangle, \langle u, v\rangle)$ -cut method, the representation of membership functions is

$$
\begin{aligned}
R_{\mathfrak{M}^{\tau}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\mathfrak{M}^{\tau}_{\mathfrak{R}}(s)} + \mathfrak{M}^{\bar{\tau}}_{\mathfrak{R}}(s)\right) ds\\
&= \frac{1}{2}\int_0^1 (\vartheta_1 + s(\vartheta_3 - \vartheta_1) + \vartheta_5 - s(\vartheta_5 - \vartheta_3)) ds\\
&= \frac{1}{2}\left[\vartheta_1 + \frac{1}{2}(\vartheta_3 - \vartheta_1) + \vartheta_5 - \frac{1}{2}(\vartheta_5 - \vartheta_3)\right]\\
&= \frac{\vartheta_1 + 2\vartheta_3 + \vartheta_5}{4},\\
R_{\alpha}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\alpha(u)} + \bar{\alpha}(u)\right) ds\\
&= \frac{1}{2}\int_0^1 \left(\vartheta_2' + u\left(\vartheta_3 - \vartheta_2'\right) + \vartheta_4' - u\left(\vartheta_4' - \vartheta_3\right)\right) du\\
&= \frac{1}{2}\left[\vartheta_2' + \frac{1}{2}\left(\vartheta_3 - \vartheta_2'\right) + \vartheta_4' - \frac{1}{2}\left(\vartheta_4' - \vartheta_3\right)\right]\\
&= \frac{\vartheta_2' + 2\vartheta_3 + \vartheta_4'}{4}.
\end{aligned}
\tag{24}
$$

Now, by the mean of $(\langle s, t\rangle, \langle u, v\rangle)$ -cut method, the representation of nonmembership functions is

$$
\begin{aligned}
R_{\mathfrak{N}^{\nu}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\mathfrak{N}^{\nu}_{\mathfrak{R}}(t)} + \mathfrak{N}^{\bar{\nu}}_{\mathfrak{R}}(t)\right) dt\\
&= \frac{1}{2}\int_0^1 (\vartheta_3 - t(\vartheta_3 - \vartheta_2) + \vartheta_3 + t(\vartheta_4 - \vartheta_3)) dt\\
&= \frac{1}{2}\left[2\vartheta_3 - \frac{1}{2}(\vartheta_3 - \vartheta_2) + \frac{1}{2}(\vartheta_4 - \vartheta_3)\right]\\
&= \frac{\vartheta_2 + 2\vartheta_3 + \vartheta_4}{4},\\
R_{\beta}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{1}{2}\int_0^1 \left(\underline{\beta(v)} + \bar{\beta}(v)\right) dv\\
&= \frac{1}{2}\int_0^1 \left(\vartheta_3 - v\left(\vartheta_3 - \vartheta_1'\right) + \vartheta_3 + v\left(\vartheta_5' - \vartheta_3\right)\right)' dv\\
&= \frac{1}{2}\left[2\vartheta_3 - \frac{1}{2}\left(\vartheta_3 - \vartheta_1'\right) + \frac{1}{2}\left(\vartheta_5' - \vartheta_3\right)\right]\\
&= \frac{\vartheta_1' + 2\vartheta_3 + \vartheta_5'}{4}.
\end{aligned}
\tag{25}
$$

Now,

$$
\begin{aligned}
R\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) &= \frac{R_{\mathfrak{M}^{\tau}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) + R_{\alpha}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) + R_{\mathfrak{N}^{\nu}_{\mathfrak{R}}}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right) + R_{\beta}\left(\pounds_{\mathfrak{R}_{\text{TLDFN}}}\right)}{4}\\
&= \frac{((\vartheta_1 + 2\vartheta_3 + \vartheta_5)/4) + \left(\left(\vartheta_2' + 2\vartheta_3 + \vartheta_4'\right)/4\right) + ((\vartheta_2 + 2\vartheta_3 + \vartheta_4)/4) + \left(\left(\vartheta_1' + 2\vartheta_3 + \vartheta_5'\right)/4\right)}{4}\\
&= \frac{\vartheta_3}{2} + \frac{\vartheta_1 + \vartheta_2 + \vartheta_4 + \vartheta_5 + \vartheta_1' + \vartheta_2' + \vartheta_4' + \vartheta_5'}{16}.
\end{aligned}
\tag{26}
$$

Consider two positive TLDFNs $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ and $\xi_{\mathfrak{R}_{\text{TLDFN}}} = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta_1', \delta_2', \delta_3, \delta_4', \delta_5') \end{cases}$, then

(i) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \prec \xi_{\mathfrak{R}_{\text{TLDFN}}}$ iff $R(\pounds_{\mathfrak{R}_{\text{TLDFN}}}) < R(\xi_{\mathfrak{R}_{\text{TLDFN}}})$

(ii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} \succ \xi_{\mathfrak{R}_{\text{TLDFN}}}$ iff $R(\pounds_{\mathfrak{R}_{\text{TLDFN}}}) > R(\xi_{\mathfrak{R}_{\text{TLDFN}}})$

(iii) $\pounds_{\mathfrak{R}_{\text{TLDFN}}} = \xi_{\mathfrak{R}_{\text{TLDFN}}}$ iff $R(\pounds_{\mathfrak{R}_{\text{TLDFN}}}) = R(\xi_{\mathfrak{R}_{\text{TLDFN}}})$

5. Solution of LDF Equations

5.1. Solution of $A + X = B$ by Using the Method of $(\langle s, t\rangle, \langle u, v\rangle)$-Cut. Let A, B, and X be the LDFNs and let $A = \begin{cases} (\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) \\ (\vartheta_1', \vartheta_2', \vartheta_3, \vartheta_4', \vartheta_5') \end{cases}$ and $B = \begin{cases} (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) \\ (\delta_1', \delta_2', \delta_3, \delta_4', \delta_5') \end{cases}$. Then,

$$
A + X = B \tag{27}
$$

is a LDF equation (LDFE). Let $X \approx \begin{cases} (x_1, x_2, x_3, x_4, x_5) \\ (x_1', x_2', x_3, x_4', x_5') \end{cases}$. Then, $X = B - A$ in general is not the solution of Equation (27).

Let

$$
\begin{aligned}
A^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{A}(s)}, \mathfrak{M}^{\bar{\tau}}_{A}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{A}(t)}, \mathfrak{N}^{\bar{\nu}}_{A}(t)\right]\right), \\ \left(\left[\underline{\alpha_A(u)}, \alpha_A\bar{(u)}\right], \left[\underline{\beta_A(v)}, \beta_A\bar{(v)}\right]\right), \end{cases}\\
B^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{B}(s)}, \mathfrak{M}^{\bar{\tau}}_{B}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{B}(t)}, \mathfrak{N}^{\bar{\nu}}_{B}(t)\right]\right), \\ \left(\left[\underline{\alpha_B(u)}, \alpha_B\bar{(u)}\right], \left[\underline{\beta_B(v)}, \beta_B\bar{(v)}\right]\right), \end{cases}\\
X^{\langle s,t\rangle}_{\langle u,v\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}^{\tau}_{X}(s)}, \mathfrak{M}^{\bar{\tau}}_{X}(s)\right], \left[\underline{\mathfrak{N}^{\nu}_{X}(t)}, \mathfrak{N}^{\bar{\nu}}_{X}(t)\right]\right), \\ \left(\left[\underline{\alpha_X(u)}, \alpha_X\bar{(u)}\right], \left[\underline{\beta_X(v)}, \beta_X\bar{(v)}\right]\right) \end{cases}
\end{aligned}
\tag{28}
$$

represent the $(\langle s, t\rangle, \langle u, v\rangle)$-cuts of A, B, and X, respectively, in the given (27). Substituting these into Equation (27), we get

$$
A^{\langle s,t\rangle}_{\langle u,v\rangle} + X^{\langle s,t\rangle}_{\langle u,v\rangle} = B^{\langle s,t\rangle}_{\langle u,v\rangle}. \tag{29}
$$

By comparing the $(\langle s, t\rangle, \langle u, v\rangle)$ -cuts of A, B, and X, we get

$$\begin{aligned}
&\left[\underline{\mathfrak{M}_A^{\tau}(s)}, \mathfrak{M}_A^{\overline{\tau}}(s)\right] + \left[\underline{\mathfrak{M}_X^{\tau}(s)}, \mathfrak{M}_X^{\overline{\tau}}(s)\right] = \left[\underline{\mathfrak{M}_B^{\tau}(s)}, \mathfrak{M}_B^{\overline{\tau}}(s)\right],\\
&\left[\underline{\mathfrak{N}_A^{\nu}(t)}, \mathfrak{N}_A^{\overline{\nu}}(t)\right] + \left[\underline{\mathfrak{N}_X^{\nu}(t)}, \mathfrak{N}_X^{\overline{\nu}}(t)\right] = \left[\underline{\mathfrak{N}_B^{\nu}(t)}, \mathfrak{N}_B^{\overline{\nu}}(t)\right],\\
&\left[\underline{\alpha_A(u)}, \overline{\alpha}_A(u)\right] + \left[\underline{\alpha_X(u)}, \overline{\alpha}_X(u)\right] = \left[\underline{\alpha_B(u)}, \overline{\alpha}_B(u)\right],\\
&\left[\underline{\beta_A(v)}, \overline{\beta}_A(v)\right] + \left[\underline{\beta_X(v)}, \overline{\beta}_X(v)\right] = \left[\underline{\beta_B(v)}, \overline{\beta}_B(v)\right].
\end{aligned} \tag{30}$$

Now,

$$\begin{aligned}
&\underline{\mathfrak{M}_X^{\tau}(s)} = \underline{\mathfrak{M}_B^{\tau}(s)} - \underline{\mathfrak{M}_A^{\tau}(s)}, \mathfrak{M}_X^{\overline{\tau}}(s) = \mathfrak{M}_B^{\overline{\tau}}(s) - \mathfrak{M}_A^{\overline{\tau}}(s),\\
&\underline{\mathfrak{N}_X^{\nu}(t)} = \underline{\mathfrak{N}_B^{\nu}(t)} - \underline{\mathfrak{N}_A^{\nu}(t)}, \mathfrak{N}_X^{\overline{\nu}}(t) = \mathfrak{N}_B^{\overline{\nu}}(t) - \mathfrak{N}_A^{\overline{\nu}}(t),\\
&\underline{\alpha_X(u)} = \underline{\alpha_B(u)} - \underline{\alpha_A(u)}, \overline{\alpha}_X(u) = \overline{\alpha}_B(u) - \overline{\alpha}_A(u),\\
&\underline{\beta_X(v)} = \underline{\beta_B(v)} - \underline{\beta_A(v)}, \overline{\beta}_X(v) = \overline{\beta}_B(v) - \overline{\beta}_A(v).
\end{aligned} \tag{31}$$

Then, the solution of the equation $A + X = B$ exists iff

(1) $\underline{\mathfrak{M}_X^{\tau}(s)}$ is monotonically increasing in $0 \le s \le 1$

(2) $\mathfrak{M}_X^{\overline{\tau}}(s)$ is monotonically decreasing in $0 \le s \le 1$

(3) $\underline{\mathfrak{N}_X^{\nu}(t)}$ is monotonically decreasing in $0 \le t \le 1$

(4) $\mathfrak{N}_X^{\overline{\nu}}(t)$ is monotonically increasing in $0 \le t \le 1$

(5) $\underline{\alpha_X(u)}$ is monotonically increasing in $0 \le u \le 1$

(6) $\overline{\alpha}_X(u)$ is monotonically decreasing in $0 \le u \le 1$

(7) $\underline{\beta_X(v)}$ is monotonically decreasing in $0 \le v \le 1$

(8) $\overline{\beta}_X(v)$ is monotonically increasing in $0 \le v \le 1$

(9) $\underline{\mathfrak{M}_X^{\tau}(1)} = \mathfrak{M}_X^{\overline{\tau}}(1) = \underline{\mathfrak{N}_X^{\nu}(0)} = \mathfrak{N}_X^{\overline{\nu}}(0) = \underline{\alpha_X(1)} = \overline{\alpha}_X(1) = \underline{\beta_X(0)} = \overline{\beta}_X(0)$.

Example 1. Consider the equation $A + X = B$, where

$$\begin{aligned}
A &= \begin{cases} (3, 5, 7, 10, 15), \\ (2, 6, 7, 8, 11), \end{cases}\\
B &= \begin{cases} (1, 6, 11, 15, 24), \\ (3, 9, 11, 13, 22). \end{cases}
\end{aligned} \tag{32}$$

The $(\langle s, t\rangle, \langle u, v\rangle)$-cuts of A, B, and X are

$$\begin{aligned}
A_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} ([3 + 4s, 15 - 8s], [7 - 2t, 7 + 3t]), \\ ([6 + u, 8 - u], [7 - 5v, 7 + 4v]), \end{cases}\\
B_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} ([1 + 10s, 24 - 13s], [11 - 5t, 11 + 4t]), \\ ([9 + 2u, 13 - 2u], [11 - 8v, 11 + 11v]), \end{cases}\\
X_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}_X^{\tau}(s)}, \mathfrak{M}_X^{\overline{\tau}}(s)\right], \left[\underline{\mathfrak{N}_X^{\nu}(t)}, \mathfrak{N}_X^{\overline{\nu}}(t)\right]\right), \\ \left(\left[\underline{\alpha_X(u)}, \overline{\alpha}_X(u)\right], \left[\underline{\beta_X(v)}, \overline{\beta}_X(v)\right]\right), \end{cases}
\end{aligned} \tag{33}$$

respectively. The $(\langle s, t\rangle, \langle u, v\rangle)$-cut equation is

$$A_{\langle u,v\rangle}^{\langle s,t\rangle} + X_{\langle u,v\rangle}^{\langle s,t\rangle} = B_{\langle u,v\rangle}^{\langle s,t\rangle}. \tag{34}$$

By comparing the $(\langle s, t\rangle, \langle u, v\rangle)$-cuts of A, B, and X, we get

$$\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(s)} &= \underline{\mathfrak{M}_B^{\tau}(s)} - \underline{\mathfrak{M}_A^{\tau}(s)} = -2 + 6s,\\
\mathfrak{M}_X^{\overline{\tau}}(s) &= \mathfrak{M}_B^{\overline{\tau}}(s) - \mathfrak{M}_A^{\overline{\tau}}(s) = 9 - 5s,\\
\underline{\mathfrak{N}_X^{\nu}(t)} &= \underline{\mathfrak{N}_B^{\nu}(t)} - \underline{\mathfrak{N}_A^{\nu}(t)} = 4 - 3t,\\
\mathfrak{N}_X^{\overline{\nu}}(t) &= \mathfrak{N}_B^{\overline{\nu}}(t) - \mathfrak{N}_A^{\overline{\nu}}(t) = 4 + t,\\
\underline{\alpha_X(u)} &= \underline{\alpha_B(u)} - \underline{\alpha_A(u)} = 3 + u,\\
\overline{\alpha}_X(u) &= \overline{\alpha}_B(u) - \overline{\alpha}_A(u) = 5 - u,\\
\underline{\beta_X(v)} &= \underline{\beta_B(v)} - \underline{\beta_A(v)} = 4 - 3v,\\
\overline{\beta}_X(v) &= \overline{\beta}_B(v) - \overline{\beta}_A(v) = 4 + 7v.
\end{aligned} \tag{35}$$

It is easy to see that $\underline{\mathfrak{M}_X^{\tau}(s)}$, $\mathfrak{N}_X^{\overline{\nu}}(t)$, $\underline{\alpha_X(u)}$, and $\overline{\beta}_X(v)$ are increasing and $\mathfrak{M}_X^{\overline{\tau}}(s)$, $\underline{\mathfrak{N}_X^{\nu}(t)}$, $\overline{\alpha}_X(u)$, and $\underline{\beta_X(v)}$ are decreasing in $0 \le s, t, u, v \le 1$. Also,

$$\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(1)} &= \mathfrak{M}_X^{\overline{\tau}}(1) = \underline{\mathfrak{N}_X^{\nu}(0)} = \mathfrak{N}_X^{\overline{\nu}}(0) = \underline{\alpha_X(1)}\\
&= \overline{\alpha}_X(1) = \underline{\beta_X(0)} = \overline{\beta}_X(0) = 4.
\end{aligned} \tag{36}$$

This shows that the solution of $A + X = B$ exists with $(\langle s, t\rangle, \langle u, v\rangle)$ -cut. The solution is

$$X = \begin{cases} (-2, 1, 4, 5, 9), \\ (1, 3, 4, 5, 11). \end{cases} \tag{37}$$

The solution in continuous form is

$$\mathfrak{M}_{\mathfrak{R}}^{\tau}(x) = \begin{cases} \dfrac{2 + x}{6}, & -2 \le x \le 4, \\ \dfrac{9 - x}{5}, & 4 \le x \le 9, \\ 0, & \text{otherwise}, \end{cases}$$

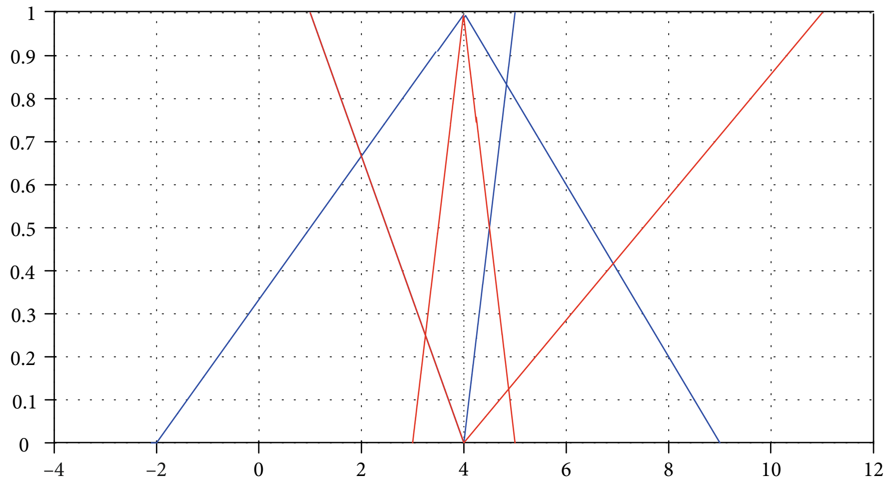

Figure 4: The graph of the solution obtained in Example 1.

$$\mathfrak{N}^{\nu}_{\mathfrak{R}}(x)=\begin{cases}\dfrac{4-x}{3}, & 1\le x\le 4,\\ -4+x, & 4\le x\le 5,\\ 0, & \text{otherwise},\end{cases}$$

$$\alpha(x)=\begin{cases}-3+x, & 3\le x\le 4,\\ 5-x, & 4\le x\le 5,\\ 0, & \text{otherwise},\end{cases}$$

$$\beta(x)=\begin{cases}\dfrac{4-x}{3}, & 1\le x\le 4,\\ \dfrac{-4+x}{7}, & 4\le x\le 11,\\ 0, & \text{otherwise}.\end{cases} \quad (38)$$

The graph of the solution is given in Figure 4.

5.2. Solution of $A\cdot X+B=C$ by Using the Method of $(\langle s,t\rangle, \langle u,v\rangle)$-Cut. Let A, B, C, and X be the LDFNs and let $A=\begin{cases}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5)\\(\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5')\end{cases}$, $B=\begin{cases}(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5)\\(\delta_1',\delta_2',\delta_3,\delta_4',\delta_5')\end{cases}$, and $C=\begin{cases}(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)\\(\eta_1',\eta_2',\eta_3,\eta_4',\eta_5')\end{cases}$. Then,

$$A\cdot X+B=C \quad (39)$$

is a LDF equation (LDFE). Let $X\approx\begin{cases}(x_1,x_2,x_3,x_4,x_5)\\(x_1',x_2',x_3,x_3',x_5')\end{cases}$. Then, $X=(C-B)/A$ in general is not the solution of Equation (39).

Let

$$\begin{aligned}
A^{\langle s,t\rangle}_{\langle u,v\rangle}&=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_A(s)},\mathfrak{M}^{\bar{\tau}}_A(s)\right],\left[\underline{\mathfrak{N}^{\nu}_A(t)},\mathfrak{N}^{\bar{\nu}}_A(t)\right]\right),\\ \left(\left[\underline{\alpha_A(u)},\bar{\alpha}_A(u)\right],\left[\underline{\beta_A(v)},\bar{\beta}_A(v)\right]\right),\end{cases}\\
B^{\langle s,t\rangle}_{\langle u,v\rangle}&=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_B(s)},\mathfrak{M}^{\bar{\tau}}_B(s)\right],\left[\underline{\mathfrak{N}^{\nu}_B(t)},\mathfrak{N}^{\bar{\nu}}_B(t)\right]\right),\\ \left(\left[\underline{\alpha_B(u)},\bar{\alpha}_B(u)\right],\left[\underline{\beta_B(v)},\bar{\beta}_B(v)\right]\right),\end{cases}\\
C^{\langle s,t\rangle}_{\langle u,v\rangle}&=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_C(s)},\mathfrak{M}^{\bar{\tau}}_C(s)\right],\left[\underline{\mathfrak{N}^{\nu}_C(t)},\mathfrak{N}^{\bar{\nu}}_C(t)\right]\right),\\ \left(\left[\underline{\alpha_C(u)},\bar{\alpha}_C(u)\right],\left[\underline{\beta_C(v)},\bar{\beta}_C(v)\right]\right),\end{cases}\\
X^{\langle s,t\rangle}_{\langle u,v\rangle}&=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_X(s)},\mathfrak{M}^{\bar{\tau}}_X(s)\right],\left[\underline{\mathfrak{N}^{\nu}_X(t)},\mathfrak{N}^{\bar{\nu}}_X(t)\right]\right),\\ \left(\left[\underline{\mathfrak{M}^{\tau}_X(s)},\mathfrak{M}^{\bar{\tau}}_X(s)\right],\left[\underline{\mathfrak{N}^{\nu}_X(t)},\mathfrak{N}^{\bar{\nu}}_X(t)\right]\right)\end{cases}
\end{aligned} \quad (40)$$

represent the $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X, respectively, in the given (39). Substituting these into Equation (39), we get

$$A^{\langle s,t\rangle}_{\langle u,v\rangle}\cdot X^{\langle s,t\rangle}_{\langle u,v\rangle}+B^{\langle s,t\rangle}_{\langle u,v\rangle}=C^{\langle s,t\rangle}_{\langle u,v\rangle}. \quad (41)$$

By comparing the $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X, we get

$$\begin{aligned}
&\left[\underline{\mathfrak{M}^{\tau}_A(s)},\mathfrak{M}^{\bar{\tau}}_A(s)\right]\cdot\left[\underline{\mathfrak{M}^{\tau}_X(s)},\mathfrak{M}^{\bar{\tau}}_X(s)\right]+\left[\underline{\mathfrak{M}^{\tau}_B(s)},\mathfrak{M}^{\bar{\tau}}_B(s)\right]=\left[\underline{\mathfrak{M}^{\tau}_C(s)},\mathfrak{M}^{\bar{\tau}}_C(s)\right],\\
&\left[\underline{\mathfrak{N}^{\nu}_A(t)},\mathfrak{N}^{\bar{\nu}}_A(t)\right]\cdot\left[\underline{\mathfrak{N}^{\nu}_X(t)},\mathfrak{N}^{\bar{\nu}}_X(t)\right]+\left[\underline{\mathfrak{N}^{\nu}_B(t)},\mathfrak{N}^{\bar{\nu}}_B(t)\right]=\left[\underline{\mathfrak{N}^{\nu}_C(t)},\mathfrak{N}^{\bar{\nu}}_C(t)\right],\\
&\left[\underline{\alpha_A(u)},\bar{\alpha}_A(u)\right]\cdot\left[\underline{\alpha_X(u)},\bar{\alpha}_X(u)\right]+\left[\underline{\alpha_B(u)},\bar{\alpha}_B(u)\right]=\left[\underline{\alpha_C(u)},\bar{\alpha}_C(u)\right],\\
&\left[\underline{\beta_A(v)},\bar{\beta}_A(v)\right]\cdot\left[\underline{\beta_X(v)},\bar{\beta}_X(v)\right]+\left[\underline{\beta_B(v)},\bar{\beta}_B(v)\right]=\left[\underline{\beta_C(v)},\bar{\beta}_C(v)\right].
\end{aligned} \quad (42)$$

Now,

$$\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(s)} &= \frac{\underline{\mathfrak{M}_C^{\tau}(s)} - \underline{\mathfrak{M}_B^{\tau}(s)}}{\underline{\mathfrak{M}_A^{\tau}(s)}},\\
\mathfrak{M}_X^{\bar{\tau}}(s) &= \frac{\mathfrak{M}_C^{\bar{\tau}}(s) - \mathfrak{M}_B^{\bar{\tau}}(s)}{\mathfrak{M}_A^{\bar{\tau}}(s)},\\
\underline{\mathfrak{N}_X^{\nu}(t)} &= \frac{\underline{\mathfrak{N}_C^{\nu}(t)} - \underline{\mathfrak{N}_B^{\nu}(t)}}{\underline{\mathfrak{N}_A^{\nu}(t)}},\\
\mathfrak{N}_X^{\bar{\nu}}(t) &= \frac{\mathfrak{N}_C^{\bar{\nu}}(t) - \mathfrak{N}_B^{\bar{\nu}}(t)}{\mathfrak{N}_A^{\bar{\nu}}(t)},\\
\underline{\alpha_X(u)} &= \frac{\underline{\alpha_C(u)} - \underline{\alpha_B(u)}}{\underline{\alpha_A(u)}},\\
\alpha_X\bar{(u)} &= \frac{\alpha_C\bar{(u)} - \alpha_B\bar{(u)}}{\alpha_A\bar{(u)}},\\
\underline{\beta_X(v)} &= \frac{\underline{\beta_C(v)} - \underline{\beta_B(v)}}{\underline{\beta_A(v)}},\\
\beta_X\bar{(v)} &= \frac{\beta_C\bar{(v)} - \beta_B\bar{(v)}}{\beta_A\bar{(v)}}.
\end{aligned} \tag{43}$$

Then, the solution of the equation $A \cdot X + B = C$ exists iff

(1) $\underline{\mathfrak{M}_X^{\tau}(s)}$ is monotonically increasing in $0 \le s \le 1$

(2) $\mathfrak{M}_X^{\bar{\tau}}(s)$ is monotonically decreasing in $0 \le s \le 1$

(3) $\underline{\mathfrak{N}_X^{\nu}(t)}$ is monotonically decreasing in $0 \le t \le 1$

(4) $\mathfrak{N}_X^{\bar{\nu}}(t)$ is monotonically increasing in $0 \le t \le 1$

(5) $\underline{\alpha_X(u)}$ is monotonically increasing in $0 \le u \le 1$

(6) $\alpha_X\bar{(u)}$ is monotonically decreasing in $0 \le u \le 1$

(7) $\underline{\beta_X(v)}$ is monotonically decreasing in $0 \le v \le 1$

(8) $\beta_X\bar{(v)}$ is monotonically increasing in $0 \le v \le 1$

$$\underline{\mathfrak{M}_X^{\tau}(1)} = \mathfrak{M}_X^{\bar{\tau}}(1) = \underline{\mathfrak{N}_X^{\nu}(0)} = \mathfrak{N}_X^{\bar{\nu}}(0) = \underline{\alpha_X(1)} = \alpha_X\bar{(1)} = \underline{\beta_X(0)} = \beta_X\bar{(0)}. \tag{44}$$

Example 2. Consider the equation $A \cdot X + B = C$, where

$$\begin{aligned}
A &= \begin{cases} (1,2,5,7,10), \\ (1,3,5,6,11), \end{cases}\\
B &= \begin{cases} (4,6,8,10,15), \\ (4,5,8,11,19), \end{cases}\\
C &= \begin{cases} (1,4,18,38,65), \\ (1,5,18,29,85). \end{cases}
\end{aligned} \tag{45}$$

The $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X are

$$\begin{aligned}
A_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} ([1+4s,10-5s],[5-3t,5+2t]), \\ ([3+2u,6-u],[5-4v,5+6v]), \end{cases}\\
B_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} ([4+4s,15-7s],[8-2t,8+2t]), \\ ([5+3u,11-3u],[8-4v,8+11v]), \end{cases}\\
C_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} ([1+17s,65-47s],[18-14t,18+20t]), \\ ([5+13u,29-11u],[18-17v,18+67v]), \end{cases}\\
X_{\langle u,v\rangle}^{\langle s,t\rangle} &= \begin{cases} \left(\left[\underline{\mathfrak{M}_X^{\tau}(s)}, \mathfrak{M}_X^{\bar{\tau}}(s)\right], \left[\underline{\mathfrak{N}_X^{\nu}(t)}, \mathfrak{N}_X^{\bar{\nu}}(t)\right]\right), \\ \left(\left[\underline{\alpha_X(u)}, \alpha_X\bar{(u)}\right], \left[\underline{\beta_X(v)}, \beta_X\bar{(v)}\right]\right), \end{cases}
\end{aligned} \tag{46}$$

respectively. The $(\langle s,t\rangle, \langle u,v\rangle)$-cut equation is

$$A_{\langle u,v\rangle}^{\langle s,t\rangle} \cdot X_{\langle u,v\rangle}^{\langle s,t\rangle} + B_{\langle u,v\rangle}^{\langle s,t\rangle} = C_{\langle u,v\rangle}^{\langle s,t\rangle}. \tag{47}$$

By comparing the $(\langle s,t\rangle, \langle u,v\rangle)$-cuts of A, B, C, and X, we get

$$\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(s)} &= \frac{(1+17s)-(4+4s)}{1+4s} = \frac{-3+13s}{1+4s},\\
\mathfrak{M}_X^{\bar{\tau}}(s) &= \frac{(65-47s)-(15-7s)}{10-5s} = \frac{10-8s}{2-s},\\
\underline{\mathfrak{N}_X^{\nu}(t)} &= \frac{(18-14t)-(8-2t)}{5-3t} = \frac{10-12t}{5-3t},\\
\mathfrak{N}_X^{\bar{\nu}}(t) &= \frac{(18+20t)-(8+2t)}{5+2t} = \frac{10+18t}{5+2t},\\
\underline{\alpha_X(u)} &= \frac{(5+13u)-(5+3u)}{3+2u} = \frac{10u}{3+2u},\\
\alpha_X\bar{(u)} &= \frac{(29-11u)-(11-3u)}{6-u} = \frac{18-8u}{6-u},\\
\underline{\beta_X(v)} &= \frac{(18-17v)-(8-4v)}{5-4v} = \frac{10-13v}{5-4v},\\
\beta_X\bar{(v)} &= \frac{(18+67v)-(8+11v)}{5+6v} = \frac{10+56v}{5+6v}.
\end{aligned} \tag{48}$$

It is easy to see that $\underline{\mathfrak{M}_X^{\tau}(s)}, \mathfrak{N}_X^{\bar{\nu}}(t), \underline{\alpha_X(u)}$, and $\beta_X\bar{(v)}$ are increasing and $\mathfrak{M}_X^{\bar{\tau}}(s), \underline{\mathfrak{N}_X^{\nu}(t)}, \alpha_X\bar{(u)}$, and $\underline{\beta_X(v)}$ are decreasing in $0 \le s, t, u, v \le 1$. Also,

$$\begin{aligned}
\underline{\mathfrak{M}_X^{\tau}(1)} &= \mathfrak{M}_X^{\bar{\tau}}(1) = \underline{\mathfrak{N}_X^{\nu}(0)} = \mathfrak{N}_X^{\bar{\nu}}(0) = \underline{\alpha_X(1)} = \alpha_X\bar{(1)}\\
&= \underline{\beta_X(0)} = \beta_X\bar{(0)} = 2.
\end{aligned} \tag{49}$$

This shows that the solution of $A \cdot X + B = C$ exists with $(\langle s,t\rangle, \langle u,v\rangle)$ -cut. The solution is

$$X = \begin{cases} (-3,-1,2,4,5), \\ (-3,0,2,3,6). \end{cases} \tag{50}$$

The solution in continuous form is

$$\mathfrak{M}^{\tau}_{\mathfrak{R}}(x)=\begin{cases}\dfrac{3+x}{13-4x}, & -3\leq x\leq 2,\\ \dfrac{-10+2x}{-8+x}, & 2\leq x\leq 5,\\ 0, & \text{otherwise},\end{cases}$$
$$\mathfrak{N}^{\nu}_{\mathfrak{R}}(x)=\begin{cases}\dfrac{-10+5x}{-12+3x}, & -1\leq x\leq 2,\\ \dfrac{-10+5x}{18-2x}, & 2\leq x\leq 4,\\ 0, & \text{otherwise},\end{cases}$$
$$\alpha(x)=\begin{cases}\dfrac{3x}{10-2x}, & 0\leq x\leq 2,\\ \dfrac{-18+6x}{-8+x}, & 2\leq x\leq 3,\\ 0, & \text{otherwise},\end{cases}$$
$$\beta(x)=\begin{cases}\dfrac{-10+5x}{-13+4x}, & -3\leq x\leq 2,\\ \dfrac{-10+5x}{56-6x}, & 2\leq x\leq 6,\\ 0, & \text{otherwise}.\end{cases}\tag{51}$$

The graph of the solution is given in Figure 5.

5.3. Solution of $A\cdot X^2+B\cdot X+C=D$ by Using the Method of α-Cut. Let A, B, C, D, and X be the LDFNs and let

$$A=\begin{cases}(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5),\\ \left(\vartheta_1',\vartheta_2',\vartheta_3,\vartheta_4',\vartheta_5'\right),\end{cases}$$
$$B=\begin{cases}(\delta_1,\delta_2,\delta_3,\delta_4,\delta_5),\\ \left(\delta_1',\delta_2^1,\delta_3,\delta_4',\delta_5'\right),\end{cases}$$
$$C=\begin{cases}(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5),\\ \left(\eta_1',\eta_2',\eta_3,\eta_4',\eta_5'\right),\end{cases}$$
$$D=\begin{cases}(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5),\\ \left(\zeta_1',\zeta_2',\zeta_3,\zeta_4',\zeta_5'\right).\end{cases}\tag{52}$$

Then,

$$A\cdot X^2+B\cdot X+C=D\tag{53}$$

is a LDF equation (LDFE). Let $X\approx\begin{cases}(x_1,x_2,x_3,x_4,x_5)\\ (x_1',x_2',x_3,x_4',x_5')\end{cases}$. Let

$$A^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_A}(s),\overline{\mathfrak{M}^{\tau}_A}(s)\right],\left[\underline{\mathfrak{N}^{\nu}_A}(t),\overline{\mathfrak{N}^{\nu}_A}(t)\right]\right),\\ \left(\left[\underline{\alpha_A}(u),\overline{\alpha_A}(u)\right],\left[\underline{\beta_A}(v),\overline{\beta_A}(v)\right]\right),\end{cases}$$
$$B^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_B}(s),\overline{\mathfrak{M}^{\tau}_B}(s)\right],\left[\underline{\mathfrak{N}^{\nu}_B}(t),\overline{\mathfrak{N}^{\nu}_B}(t)\right]\right),\\ \left(\left[\underline{\alpha_B}(u),\overline{\alpha_B}(u)\right],\left[\underline{\beta_B}(v),\overline{\beta_B}(v)\right]\right),\end{cases}$$
$$C^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_C}(s),\overline{\mathfrak{M}^{\tau}_C}(s)\right],\left[\underline{\mathfrak{N}^{\nu}_C}(t),\overline{\mathfrak{N}^{\nu}_C}(t)\right]\right),\\ \left(\left[\underline{\alpha_C}(u),\overline{\alpha_C}(u)\right],\left[\underline{\beta_C}(v),\overline{\beta_C}(v)\right]\right),\end{cases}$$
$$D^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_D}(s),\overline{\mathfrak{M}^{\tau}_D}(s)\right],\left[\underline{\mathfrak{N}^{\nu}_D}(t),\overline{\mathfrak{N}^{\nu}_D}(t)\right]\right),\\ \left(\left[\underline{\alpha_D}(u),\overline{\alpha_D}(u)\right],\left[\underline{\beta_D}(v),\overline{\beta_D}(v)\right]\right),\end{cases}$$
$$X^{\langle s,t\rangle}_{\langle u,v\rangle}=\begin{cases}\left(\left[\underline{\mathfrak{M}^{\tau}_X}(s),\overline{\mathfrak{M}^{\tau}_X}(s)\right],\left[\underline{\mathfrak{N}^{\nu}_X}(t),\overline{\mathfrak{N}^{\nu}_X}(t)\right]\right),\\ \left(\left[\underline{\alpha_X}(u),\overline{\alpha_X}(u)\right],\left[\underline{\beta_X}(v),\overline{\beta_X}(v)\right]\right)\end{cases}\tag{54}$$

represent the $(\langle s,t\rangle,\langle u,v\rangle)$-cuts of A, B, C, D, and X, respectively, in the given (53). Substituting these into Equation (53), we get

$$A^{\langle s,t\rangle}_{\langle u,v\rangle}\cdot\left(X^{\langle s,t\rangle}_{\langle u,v\rangle}\right)^2+B^{\langle s,t\rangle}_{\langle u,v\rangle}\cdot X^{\langle s,t\rangle}_{\langle u,v\rangle}+C^{\langle s,t\rangle}_{\langle u,v\rangle}=D^{\langle s,t\rangle}_{\langle u,v\rangle}.\tag{55}$$

By comparing the $(\langle s,t\rangle,\langle u,v\rangle)$ -cuts of A, B, C, D, and X, we get

$$\begin{cases}\left[\underline{\mathfrak{M}^{\tau}_A}(s),\overline{\mathfrak{M}^{\tau}_A}(s)\right]\cdot\left[\underline{\mathfrak{M}^{\tau}_X}(s),\overline{\mathfrak{M}^{\tau}_X}(s)\right]^2+\left[\underline{\mathfrak{M}^{\tau}_B}(s),\overline{\mathfrak{M}^{\tau}_B}(s)\right]\\ \quad\cdot\left[\underline{\mathfrak{M}^{\tau}_X}(s),\overline{\mathfrak{M}^{\tau}_X}(s)\right]=\left[\underline{\mathfrak{M}^{\tau}_D}(s),\overline{\mathfrak{M}^{\tau}_D}(s)\right]+\left[\underline{\mathfrak{M}^{\tau}_C}(s),\overline{\mathfrak{M}^{\tau}_C}(s)\right],\end{cases}$$
$$\begin{cases}\left[\underline{\mathfrak{N}^{\nu}_A}(t),\overline{\mathfrak{N}^{\nu}_A}(t)\right]\cdot\left[\underline{\mathfrak{N}^{\nu}_X}(t),\overline{\mathfrak{N}^{\nu}_X}(t)\right]^2+\left[\underline{\mathfrak{N}^{\nu}_B}(t),\overline{\mathfrak{N}^{\nu}_B}(t)\right]\\ \quad\cdot\left[\underline{\mathfrak{N}^{\nu}_X}(t),\overline{\mathfrak{N}^{\nu}_X}(t)\right]=\left[\underline{\mathfrak{N}^{\nu}_D}(t),\overline{\mathfrak{N}^{\nu}_D}(t)\right]+\left[\underline{\mathfrak{N}^{\nu}_C}(t),\overline{\mathfrak{N}^{\nu}_C}(t)\right],\end{cases}$$
$$\begin{cases}\left[\underline{\alpha_A}(u),\overline{\alpha_A}(u)\right]\cdot\left[\underline{\alpha_X}(u),\overline{\alpha_X}(u)\right]^2+\left[\underline{\alpha_B}(u),\overline{\alpha_B}(u)\right]\\ \quad\cdot\left[\underline{\alpha_X}(u),\overline{\alpha_X}(u)\right]=\left[\underline{\alpha_D}(u),\overline{\alpha_D}(u)\right]+\left[\underline{\alpha_C}(u),\overline{\alpha_C}(u)\right],\end{cases}$$
$$\begin{cases}\left[\underline{\beta_A}(v),\overline{\beta_A}(v)\right]\cdot\left[\underline{\beta_X}(v),\overline{\beta_X}(v)\right]^2+\left[\underline{\beta_B}(v),\overline{\beta_B}(v)\right]\\ \quad\cdot\left[\underline{\beta_X}(v),\overline{\beta_X}(v)\right]=\left[\underline{\beta_D}(v),\overline{\beta_D}(v)\right]+\left[\underline{\beta_C}(v),\overline{\beta_C}(v)\right].\end{cases}\tag{56}$$

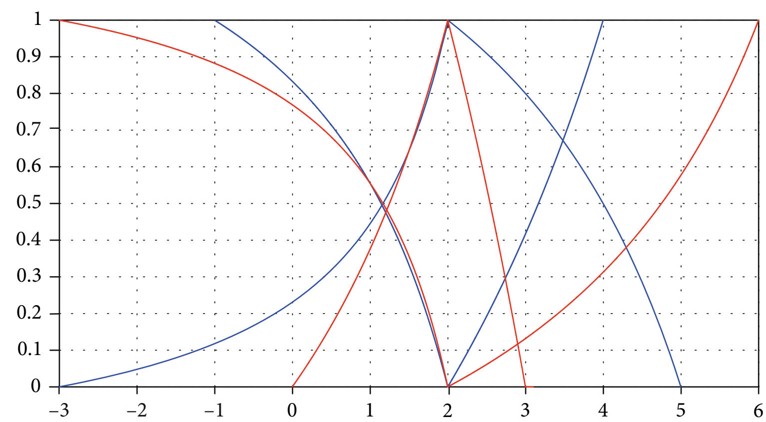

Figure 5: The graph of the solution obtained in Example 2.

Now,

$$\underline{\mathfrak{M}^{\tau}_X(s)} = \frac{-\underline{\mathfrak{M}^{\tau}_B(s)} \pm \sqrt{\underline{\mathfrak{M}^{\tau}_B(s)}^{\,2} - 4\left(\underline{\mathfrak{M}^{\tau}_A(s)}\right)\left(\underline{\mathfrak{M}^{\tau}_C(s)} - \underline{\mathfrak{M}^{\tau}_D(s)}\right)}}{2\underline{\mathfrak{M}^{\tau}_A(s)}},$$

$$\mathfrak{M}^{\overline{\tau}}_X(s) = \frac{-\mathfrak{M}^{\overline{\tau}}_B(s) \pm \sqrt{\mathfrak{M}^{\overline{\tau}}_B(s)^2 - 4\left(\mathfrak{M}^{\overline{\tau}}_A(s)\right)\left(\mathfrak{M}^{\overline{\tau}}_C(s) - \mathfrak{M}^{\overline{\tau}}_D(s)\right)}}{2\mathfrak{M}^{\overline{\tau}}_A(s)},$$

$$\underline{\mathfrak{N}^{\nu}_X(t)} = \frac{-\underline{\mathfrak{N}^{\nu}_B(t)} \pm \sqrt{\underline{\mathfrak{N}^{\nu}_B(t)}^{\,2} - 4\left(\underline{\mathfrak{N}^{\nu}_A(t)}\right)\left(\underline{\mathfrak{N}^{\nu}_C(t)} - \underline{\mathfrak{N}^{\nu}_D(t)}\right)}}{2\underline{\mathfrak{N}^{\nu}_A(t)}},$$

$$\mathfrak{N}^{\overline{\nu}}_X(t) = \frac{-\mathfrak{N}^{\overline{\nu}}_B(t) \pm \sqrt{\mathfrak{N}^{\overline{\nu}}_B(t)^2 - 4\left(\mathfrak{N}^{\overline{\nu}}_A(t)\right)\left(\mathfrak{N}^{\overline{\nu}}_C(t) - \mathfrak{N}^{\overline{\nu}}_D(t)\right)}}{2\mathfrak{N}^{\overline{\nu}}_A(t)},$$

$$\underline{\alpha_X(u)} = \frac{-\underline{\alpha_B(u)} \pm \sqrt{\underline{\alpha_B(u)}^{\,2} - 4\left(\underline{\alpha_A(u)}\right)\left(\underline{\alpha_C(u)} - \underline{\alpha_D(u)}\right)}}{2\underline{\alpha_A(u)}},$$

$$\overline{\alpha}_X(u) = \frac{-\overline{\alpha}_B(u) \pm \sqrt{\overline{\alpha}_B(u)^2 - 4(\overline{\alpha}_A(u))(\overline{\alpha}_C(u) - \overline{\alpha}_D(u))}}{2\overline{\alpha}_A(u)},$$

$$\underline{\beta_X(v)} = \frac{-\underline{\beta_B(v)} \pm \sqrt{\underline{\beta_B(v)}^{\,2} - 4\left(\underline{\beta_A(v)}\right)\left(\underline{\beta_C(v)} - \underline{\beta_D(v)}\right)}}{2\underline{\beta_A(v)}},$$

$$\overline{\beta}_X(v) = \frac{-\overline{\beta}_B(v) \pm \sqrt{\overline{\beta}_B(v)^2 - 4(\overline{\beta}_A(v))(\overline{\beta}_C(v) - \overline{\beta}_D(v))}}{2\overline{\beta}_A(v)}. \tag{57}$$

Then, the solution of the equation $A \cdot X^2 + B \cdot X + C = D$ exists iff

(1) $\underline{\mathfrak{M}^{\tau}_X(s)}$ is monotonically increasing in $0 \le s \le 1$

(2) $\mathfrak{M}^{\overline{\tau}}_X(s)$ is monotonically decreasing in $0 \le s \le 1$

(3) $\underline{\mathfrak{N}^{\nu}_X(t)}$ is monotonically decreasing in $0 \le t \le 1$

(4) $\mathfrak{N}^{\overline{\nu}}_X(t)$ is monotonically increasing in $0 \le t \le 1$

Table 1: $\langle s, t\rangle$-cuts of A, B, C, D, and X.

$\langle s, t\rangle$-cuts	X	A	B	C	D	$C-D$
$\vartheta_1 + s(\vartheta_3 - \vartheta_1)$	$\underline{\mathfrak{M}^{\tau}_X(s)}$	$4+3s$	$2+3s$	$1+3s$	$1+5s$	$-2s$
$\vartheta_5 - s(\vartheta_5 - \vartheta_3)$	$\mathfrak{M}^{\overline{\tau}}_X(s)$	$10-3s$	$8-3s$	$7-3s$	$12-6s$	$-5+3s$
$\vartheta_3 - t(\vartheta_3 - \vartheta_2)$	$\underline{\mathfrak{N}^{\nu}_X(t)}$	$7-2t$	$5-t$	$4-2t$	$6-3t$	$-2+t$
$\vartheta_3 + t(\vartheta_4 - \vartheta_3)$	$\mathfrak{N}^{\overline{\nu}}_X(t)$	$7+2t$	$5+t$	$4+t$	$6+2t$	$-2-t$

(5) $\underline{\alpha_X(u)}$ is monotonically increasing in $0 \le u \le 1$

(6) $\overline{\alpha}_X(u)$ is monotonically decreasing in $0 \le u \le 1$

(7) $\underline{\beta_X(v)}$ is monotonically decreasing in $0 \le v \le 1$

(8) $\overline{\beta}_X(v)$ is monotonically increasing in $0 \le v \le 1$

(9) $\underline{\mathfrak{M}^{\tau}_X(1)} = \mathfrak{M}^{\overline{\tau}}_X(1) = \underline{\mathfrak{N}^{\nu}_X(0)} = \mathfrak{N}^{\overline{\nu}}_X(0) = \underline{\alpha_X(1)} = \overline{\alpha}_X(1) = \underline{\beta_X(0)} = \overline{\beta}_X(0)$

Example 3. Consider the equation $A \cdot X^2 + B \cdot X + C = D$, *where*

$$\begin{aligned} A &= \begin{cases} (4, 5, 7, 9, 10), \\ (2, 6, 7, 8, 13), \end{cases} \\ B &= \begin{cases} (2, 4, 5, 6, 8), \\ (4, 4, 5, 6, 7), \end{cases} \\ C &= \begin{cases} (1, 2, 4, 5, 7), \\ (1, 3, 4, 5, 7), \end{cases} \\ D &= \begin{cases} (1, 3, 6, 8, 12), \\ (1, 4, 6, 8, 11). \end{cases} \end{aligned} \tag{58}$$

The $\langle s, t\rangle$-cuts of A, B, C, D, and X are given in Table 1.

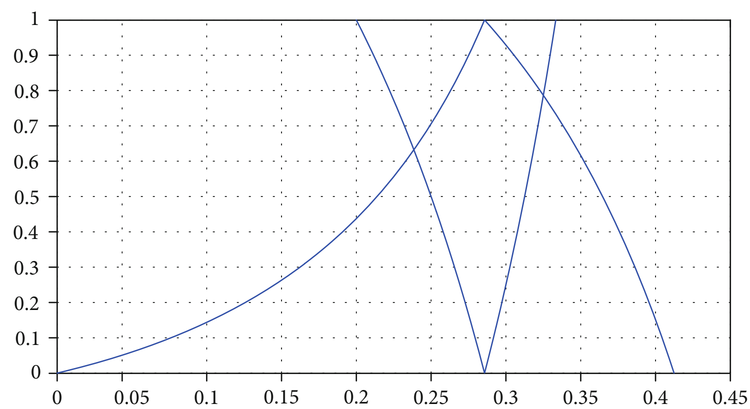

Figure 6: Solution by $\langle s, t\rangle$-cut.

By comparing the $\langle s, t\rangle$-cuts of A, B, C, D, and X, we get

$$\begin{aligned}
\underline{\mathfrak{M}^{\tau}_X(s)} &= \frac{-(2+3s)+\sqrt{(2+3s)^2-4(4+3s)(-2s)}}{2(4+3s)},\\
\mathfrak{M}^{\bar{\tau}}_X(s) &= \frac{-(8-3s)+\sqrt{(8-3s)^2-4(10-3s)(-5+3s)}}{2(10-3s)},\\
\underline{\mathfrak{N}^{\nu}_X(t)} &= \frac{-(5-t)+\sqrt{(5-t)^2-4(7-2t)(-2+t)}}{2(7-2t)},\\
\mathfrak{N}^{\bar{\nu}}_X(t) &= \frac{-(5+t)+\sqrt{(5+t)^2-4(7+2t)(-2-t)}}{2(7+2t)}.
\end{aligned} \tag{59}$$

The graph obtained by $\langle s, t\rangle$-cut is shown in Figure 6. The $\langle u, v\rangle$-cuts of A, B, C, D, and X are given in Table 2. By comparing the $\langle u, v\rangle$-cuts of A, B, C, D, and X, we get

$$\begin{aligned}
\underline{\alpha(u)} &= \frac{-(4+u)+\sqrt{(4+u)^2-4(6+u)(-1-u)}}{2(6+u)},\\
\overline{\alpha}(u) &= \frac{-(6-u)+\sqrt{(6-u)^2-4(8-u)(-3+u)}}{2(8-u)},\\
\underline{\beta(v)} &= \frac{-(5-v)+\sqrt{(5-v)^2-4(7-5v)(-2+2v)}}{2(7-5v)},\\
\overline{\beta}(v) &= \frac{-(5+2v)+\sqrt{(5+2v)^2-4(7+6v)(-2-2v)}}{2(7+6v)}.
\end{aligned} \tag{60}$$

The graph obtained by $\langle u, v\rangle$-cut is shown in Figure 7.

It is easy to see that $\underline{\mathfrak{M}^{\tau}_X(s)}$, $\mathfrak{N}^{\bar{\nu}}_X(t)$, $\underline{\alpha_X(u)}$, and $\overline{\beta}_X(v)$ are increasing and $\mathfrak{M}^{\bar{\tau}}_X(s)$, $\underline{\mathfrak{N}^{\nu}_X(t)}$, $\overline{\alpha}_X(u)$, and $\underline{\beta_X(v)}$ are decreasing in $0 \le s, t, u, v \le 1$. Also,

Table 2: $\langle u, v\rangle$-cuts of A, B, C, D, and X.

$\langle u, v\rangle$-cuts	X	A	B	C	D	$C-D$
$\vartheta_2'+u\left(\vartheta_3-\vartheta_2'\right)$	$\underline{\alpha(u)}$	$6+u$	$4+u$	$3+u$	$4+2u$	$-1-u$
$\vartheta_4'-u\left(\vartheta_4'-\vartheta_3\right)$	$\overline{\alpha}(u)$	$8-u$	$6-u$	$5-u$	$8-2u$	$-3+u$
$\vartheta_3-v\left(\vartheta_3-\vartheta_1'\right)$	$\underline{\beta(v)}$	$7-5v$	$5-v$	$4-3v$	$6-5v$	$-2+2v$
$\vartheta_3+v\left(\vartheta_5'-\vartheta_3\right)$	$\overline{\beta}(v)$	$7+6v$	$5+2v$	$4+3v$	$6+5v$	$-2-2v$

$$\begin{aligned}
\underline{\mathfrak{M}^{\tau}_X(1)} &= \mathfrak{M}^{\bar{\tau}}_X(1) = \underline{\mathfrak{N}^{\nu}_X(0)} = \mathfrak{N}^{\bar{\nu}}_X(0) = \underline{\alpha_X(1)} = \overline{\alpha}_X(1)\\
&= \underline{\beta_X(0)} = \overline{\beta}_X(0) = 0.2857.
\end{aligned} \tag{61}$$

This shows that the solution of $A\cdot X^2+B\cdot X+C=D$ exists with $(\langle s, t\rangle, \langle u, v\rangle)$ -cut. The solution is

$$X = \begin{cases} \left(0,\frac{1}{5},\frac{2}{7},\frac{1}{3},\frac{-4+\sqrt{66}}{10}\right) \\ \left(0,\frac{-2+\sqrt{10}}{6},\frac{2}{7},\frac{-3+\sqrt{33}}{8},\frac{-7+\sqrt{257}}{26}\right) \end{cases} = \begin{cases} (0,0.2,0.2857,0.3333,0.4124), \\ (0,0.1937,0.2857,0.3431,0.3474). \end{cases} \tag{62}$$

The solution in continuous form is

$$\mathfrak{M}^{\tau}_{\mathfrak{R}}(x) = \begin{cases} -\dfrac{2x(2x+1)}{3x^2+3x-2}, & 0 \le x \le 0.2857, \\ \dfrac{10x^2+8x-5}{3(x^2+x-1)}, & 0.2857 \le x \le 0.4124, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathfrak{N}^{\nu}_{\mathfrak{R}}(x) = \begin{cases} \dfrac{7x-2}{2x-1}, & 0.2 \le x \le 0.2857, \\ -\dfrac{7x-2}{2x-1}, & 0.2857 \le x \le 0.3333, \\ 0, & \text{otherwise}, \end{cases}$$

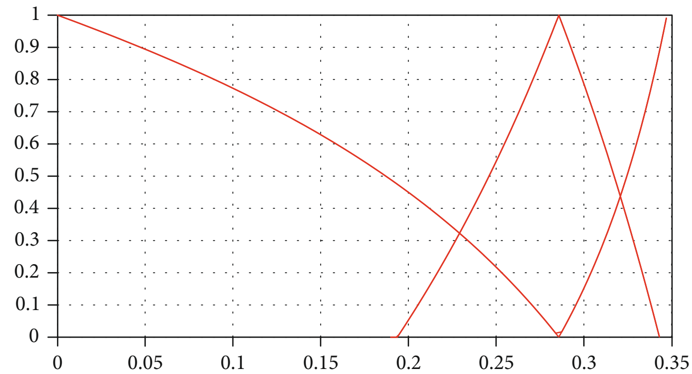

Figure 7: Solution by $\langle u, v \rangle$-cut.

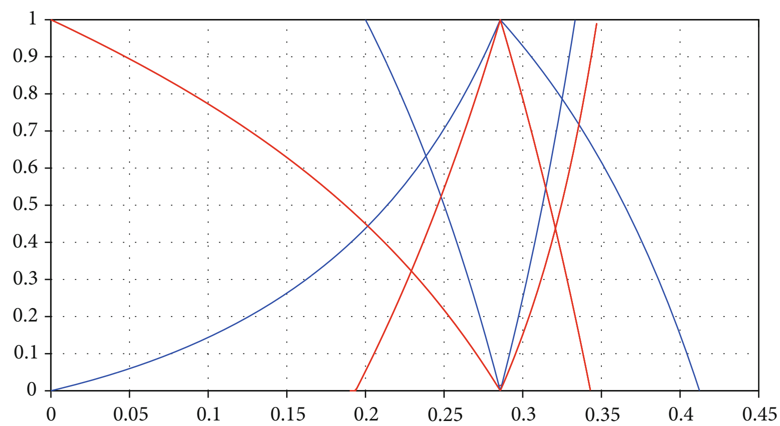

Figure 8: The graph of the solution obtained in Example 3.

$$\alpha(x) = \begin{cases} \dfrac{-6x^2 - 4x + 1}{x^2 + x - 1}, & 0.1937 \leq x \leq 0.2857, \\ \dfrac{8x^2 + 6x - 3}{x^2 + x - 1}, & 0.2857 \leq x \leq 0.3431, \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta(x) = \begin{cases} \dfrac{7x^2 + 5x - 2}{5x^2 + x - 2}, & 0 \leq x \leq 0.2857, \\ \dfrac{-7x^2 - 5x + 2}{2(3x^2 + x - 1)}, & 0.2857 \leq x \leq 0.3474, \\ 0, & \text{otherwise}. \end{cases} \quad (63)$$

The graph of the solution is given in Figure 8.

6. Conclusion

In this paper, we have defined the linear Diophantine fuzzy numbers, in particular triangular linear Diophantine fuzzy number, and present some properties related to them. After finding the ranking function of triangular linear Diophantine fuzzy number, our study has focussed on the linear Diophantine fuzzy equations. We used the more general approach to solve LDF equations that is the method of $(\langle s, t \rangle, \langle u, v \rangle)$-cut. In LDF sets, there is no limitation to take the grades like in intuitionistic fuzzy sets, Pythagorean fuzzy sets, and q-rung orthopair fuzzy sets. The linear Diophantine fuzzy numbers may have several applications, like in linear programming, transportation problems, assignment problems, and shortest route problems. Our future work may be on the following topics:

(i) LDF linear programming problems

(ii) LDF assignment problems and transportation problems

(iii) LDF shortest path problems

(iv) Numerical solutions of linear and nonlinear LDF equations

Data Availability

No data were used to support this study.

Disclosure

The statements made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors of this paper declare that they have no conflict of interest.

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