FIXED POINT RESULTS FOR A PAIR OF MULTIVALUED MAPPINGS IN QUASI METRIC SPACES VIA NEW APPROACH

Abdullah Shoaib[1], Shaif Saleh Alshoraify[2], Muhammad Arshad[3]

In this paper, our purpose is to show that multivalued mappings satisfying new generalized contraction on the intersection of an open ball and a sequence involving rational expression have a common fixed point in left K-sequentially complete quasi metric space. We consider an illustrative example. Moreover, we apply our results to obtain results endowed with a graph and an order in quasi metric spaces.

Keywords: common fixed point; left K-sequentially quasi metric space; open ball; multivalued mappings; graph; order.

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1. Introduction

Fixed point theory (see [1]-[28]) has been an active research field over the last six decades. One of the generalizations of the metric space is the quasi metric space that was introduced by Wilson [28]. The commutativity condition does not hold in general in a quasi metric space. Several authors used these concepts to prove some fixed point theorems, see [8, 14].

Nadler [15] initiated the study of fixed point theory for multivalued mappings. Since then, an interesting and rich fixed point theory for such mappings was developed in many directions, see [3, 6, 27]. Arshad et al. [4] observed that there was mappings which had fixed point but there was no any result to ensure the existence of fixed point of such mappings. They introduced a contraction on closed ball to achieve common fixed points for such mappings. For further results on closed balls, see (see [4, 5, 17, 18, 21, 25, 26]).

In this paper, we have extended the result of Altun et al. [2] in five different ways by using

(i) multivalued mappings instead of single-valued mappings;

(ii) open ball instead of whole space;

(iii) new generalized contraction instead of Banach type contraction;

(iv) left K-sequentially complete quasi metric space instead of complete metric space.

(v) generalized function $\alpha : X \times X \to [0, +\infty)$ instead of partial order relation.

We apply our result to obtain results in ordered spaces and spaces endowed with a graph. We recall the following definitions and results which will be useful to understand the paper.

Definition 1.1 [2] Let $\mu \in \Psi$ and Ψ denotes the set of functions $\mu : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Ψ_1) μ is non-decreasing.

[1]Assistant Professor, Department of Mathematics and Statistics, Riphah International University, Islamabad - 44000, Pakistan, e-mail: abdullahshoaib15@yahoo.com

[2]PhD scholar, Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad - 44000, Pakistan, e-mail: shaifsaleh21@gmail.com

[3]Professor, Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad - 44000, Pakistan, e-mail: marshadzia@iiu.edu.pk

(Ψ_2) For all $t > 0$, we have $\sum_{k=0}^{\infty} \mu^k(t) < \infty$, where μ^k is the k^{th} iterate of μ. The function $\mu \in \Psi$ is called comparison function.

Lemma 1.1 [2] Let $\mu \in \Psi$. Then

(i) $\mu(t) < t$, for all $t > 0$,

(ii) $\mu(0) = 0$.

Definition 1.3 [28] Let X be anon empty set and $q : X \times X \to [0, \infty)$ be a function, which satisfies:

(d_1) $q(x, y) = 0$ if and only if $x = y$,

(d_2) $q(x, y) \leq q(x, z) + q(z, y)$.

Then q is called a quasi metric and the pair (X, q) is called a quasi metric space. For $x \in X$ and $\varepsilon > 0$, $B_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon$ and $q(y, x) < \varepsilon\}$ and $\overline{B_q(x, \varepsilon)} = \{y \in X : q(x, y) \leq \varepsilon$ and $q(y, x) \leq \varepsilon\}$ are open ball and closed ball in (X, q) respectively.

Reilly et al. [20] introduced the notion of left (right) K -Cauchy sequence and left (right) K-sequentially complete spaces.

Definition 1.4 [20] Let (X, q) be a quasi metric space.

(a) A sequence $\{x_n\}$ in (X, q) is called left (right) K- Cauchy if for every $\varepsilon > 0$, there is exists $n_0 \in N$ such that $q(x_m, x_n) < \varepsilon$ (respectively $q(x_n, x_m) < \varepsilon$).

(b) A sequence $\{x_n\}$ in (X, q) is converges to x, if $\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x) = 0$. In this case, the point x is called a limit of the sequence $\{x_n\}$.

(c) (X, q) is called left (right) K -sequentially complete if every left (right) K -Cauchy sequence in q -converges to a point $x \in X$ such that $q(x, x) = 0$.

Definition 1.5 [27] Let (X, q) be a quasi metric space. Let K be a non empty subset of X and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$q(x, K) = q(x, y_0);\ \text{where } q(x, K) = \inf\{q(x, y),\ y \in K\},$$

$$\text{and } q(K, x) = q(y_0, x),\ \text{where } q(K, x) = \inf\{q(y, x),\ y \in K\}.$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set. We denote $P(X)$ be the set of all proximinal subsets of X.

Definition 1.6 [27] The function $H_q : P(X) \times P(X) \to X$, defined by

$$H_q(A, B) = \max\left\{\sup_{x\in A} q(x, B), \sup_{y\in B} q(y, A)\right\},$$

is called quasi Hausdorff metric on $P(X)$. Also $(P(X), H_q)$ is known as quasi Hausdorff metric space.

Lemma 1.2 [27] *Let (X, q) be a quasi metric space. Let $(P(X), H_q)$ be quasi Hausdorff metric space on $P(X)$. Then for all $A, B \in P(X)$ and for each $a \in A$ there exists $b_a \in B$, such that $q(a, b) \leq H_q(A, B)$ and $H_q(B, A) \geq q(b_a, a)$.*

Definition 1.7 Let X be a non empty set and $\alpha : X \times X \to [0, +\infty)$ be a mapping such that $\alpha(x, y) \geq 1$ and $\alpha(y, x) \geq 1$ implies $x = y$. Let $M \subseteq X$, define $\alpha^*(x, M) = \inf\{\alpha(x, a), a \in M\}$ and $\alpha^*(M, y) = \inf\{\alpha(b, y), b \in M\}$.

Lemma 1.3 [27] *Every closed set Y in a left (right) K-sequentially complete quasi metric space X is left (right) K-sequentially complete.*

2. Main result

Let (X, q) be a quasi metric space, $x_0 \in X$ and $T : X \to P(X)$ be a multivalued mapping on X. As Tx_0 is a proximinal set, then there exists $x_1 \in Tx_0$ such that $q(x_0, Tx_0) = q(x_0, x_1)$ and $q(Tx_0, x_0) = q(x_1, x_0)$. Now, for $x_1 \in X$, there exist $x_2 \in Tx_1$

such that $q(x_1, Tx_1) = q(x_1, x_2)$ and $q(Tx_1, x_1) = q(x_2, x_1)$. Continuing this process, we construct a sequence x_n of points in X such that $x_{n+1} \in Tx_n$, $q(x_n, Tx_n) = q(x_n, x_{n+1})$ and $q(Tx_n, x_n) = q(x_{n+1}, x_n)$. We denote this iterative sequence $\{XT(x_n)\}$ and say that $\{XT(x_n)\}$ is a sequence in X generated by x_0.

Theorem 2.1 *Let (X, q) be a left K-sequentially complete quasi metric space, $S, T : X \to P(X)$ be the multivalued mappings, $\mu \in \Psi$, $x_0 \in X$, $r > 0$ and $\alpha : X \times X \to [0, +\infty)$. Suppose that the following assumptions hold:*

(i) For every $x, y \in B_q(x_0, r) \cap \{XT(x_n)\}$ with $\alpha^*(Sx, x) \geq 1$, $\alpha^*(y, Sy) \geq 1$, we have

$$\max\{H_q(Tx, Ty), H_q(Ty, Tx)\} \leq \mu(P_q(x, y)), \tag{1}$$

where

$$P_q(x, y) = \max\left\{q(x, y), q(x, Tx), \frac{q(x, Tx)\, q(x, Ty) + q(y, Ty)\, q(y, Tx)}{q(x, Ty) + q(y, Tx)}\right\}.$$

(ii)

$$\sum_{i=0}^{j} \max\left\{\mu^i(q(x_1, x_0)), \mu^i(q(x_0, x_1))\right\} < r, \text{ for all } j \in N \cup \{0\}. \tag{2}$$

(iii) If $x \in B_q(x_0, r)$, $q(x, Tx) = q(x, y)$ and $q(Tx, x) = q(y, x)$, then

(a) $\alpha^*(x, Sx) \geq 1$, implies $\alpha^*(Sy, y) \geq 1$, (b) $\alpha^*(Sx, x) \geq 1$, implies $\alpha^*(y, Sy) \geq 1$.

(iv) The set $G(S) = \{x : \alpha^*(x, Sx) \geq 1 \text{ and } x \in B_q(x_0, r)\}$ is closed and contained x_0.

Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$ and a sequence $\{x_{2n}\} \to x^* \in G(S)$. Also, if inequality (1) holds for $x, y \in \{x^*\}$, then T and S have a common fixed point x^* in $B_q(x_0, r)$.

Proof. As x_0 be an arbitrary element of $G(S)$, from condition (iv) $\alpha^*(x_0, Sx_0) \geq 1$. Consider the sequence $\{XT(x_n)\}$. Then there exists $x_1 \in Tx_0$ such that

$$q(x_0, Tx_0) = q(x_0, x_1) \text{ and } q(Tx_0, x_0) = q(x_1, x_0).$$

From condition (iii) $\alpha^*(Sx_1, x_1) \geq 1$. In particular, (2) holds for $j = 0$, so

$$\max\{q(x_1, x_0), q(x_0, x_1)\} < r.$$

Therefore $q(x_1, x_0) < r$ and $q(x_0, x_1) < r$. Hence $x_1 \in B_q(x_0, r)$. Let $x_2, ...x_j \in B_q(x_0, r) \cap \{XT(x_n)\}$, $\alpha^*(x_j, Sx_j) \geq 1$ and $\alpha^*(Sx_{j+1}, x_{j+1}) \geq 1$, for some $j \in N$, where $j = 2i$, $i = 2, 3, ..., \frac{j}{2}$. Now by Lemma 1.2, we have

$$\begin{aligned} q(x_{2i}, x_{2i+1}) &\leq H_q(Tx_{2i-1}, Tx_{2i}) \\ &\leq \max\{H_q(Tx_{2i-1}, Tx_{2i}), H_q(Tx_{2i}, Tx_{2i-1})\}. \end{aligned}$$

As $x_{2i-1}, x_{2i} \in B_q(x_0, r) \cap \{XT(x_n)\}$, $\alpha^*(x_{2i}, Sx_{2i}) \geq 1$ and $\alpha^*(Sx_{2i-1}, x_{2i-1}) \geq 1$, then by (1), we have

$$\begin{aligned} q(x_{2i}, x_{2i+1}) \leq\ & \mu\Bigg(\max\Bigg\{q(x_{2i-1}, x_{2i}), q(x_{2i-1}, x_{2i}), \\ & \frac{q(x_{2i-1}, x_{2i})\, q(x_{2i-1}, Tx_{2i}) + q(x_{2i}, x_{2i+1})\, q(x_{2i}, Tx_{2i-1})}{q(x_{2i-1}, Tx_{2i}) + q(x_{2i}, Tx_{2i-1})}\Bigg\}\Bigg), \end{aligned}$$

$$q(x_{2i}, x_{2i+1}) \leq \mu(q(x_{2i-1}, x_{2i})). \tag{3}$$

Which implies that

$$q(x_{2i}, x_{2i+1}) \leq \max\{\mu(q(x_{2i-1}, x_{2i})), \mu(q(x_{2i}, x_{2i-1}))\}. \tag{4}$$

Again by Lemma 1.2, we have

$$\begin{aligned} q(x_{2i-1}, x_{2i}) &\leq H_q(Tx_{2i-2}, Tx_{2i-1}) \\ &\leq \max\{H_q(Tx_{2i-2}, Tx_{2i-1}), H_q(Tx_{2i-1}, Tx_{2i-2})\}. \end{aligned}$$

As $x_{2i-1}, x_{2i-2} \in B_q(x_0, r) \cap \{XTx_n\}$, $\alpha^*(Sx_{2i-1}, x_{2i-1}) \geq 1$ and $\alpha^*(x_{2i-2}, Sx_{2i-2}) \geq 1$, then by (1), we have

$$\begin{aligned} q(x_{2i-1}, x_{2i}) &\leq \mu(\max\{q(x_{2i-1}, x_{2i-2}), q(x_{2i-1}, x_{2i}), q(x_{2i-2}, x_{2i-1})\}) \\ &= \mu(\max\{q(x_{2i-1}, x_{2i-2}), q(x_{2i-2}, x_{2i-1})\}). \end{aligned}$$

As μ is non decreasing function, so

$$\mu(q(x_{2i-1}, x_{2i})) \leq \max\left\{\mu^2(q(x_{2i-1}, x_{2i-2})), \mu^2(q(x_{2i-2}, x_{2i-1}))\right\}. \quad (5)$$

Using (5) in (3), then

$$q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu^2(q(x_{2i-1}, x_{2i-2})), \mu^2(q(x_{2i-2}, x_{2i-1}))\right\}. \quad (6)$$

Now, by Lemma 1.2

$$q(x_{2i-2}, x_{2i-1}) \leq H_q(Tx_{2i-3}, Tx_{2i-2}).$$

As $x_{2i-3}, x_{2i-2} \in B_q(x_0, r) \cap \{XTx_n\}$, $\alpha^*(x_{2i-2}, Sx_{2i-2}) \geq 1$ and $\alpha^*(Sx_{2i-3}, x_{2i-3}) \geq 1$, then by (1), we have

$$q(x_{2i-2}, x_{2i-1}) \leq \mu(q(x_{2i-3}, x_{2i-2})). \quad (7)$$

Which implies that

$$\mu^2(q(x_{2i-2}, x_{2i-1})) \leq \mu^2(\mu(\max\{q(x_{2i-3}, x_{2i-2}), q(x_{2i-2}, x_{2i-3})\})). \quad (8)$$

Now, by Lemma 1.2

$$q(x_{2i-1}, x_{2i-2}) \leq H_q(Tx_{2i-2}, Tx_{2i-3}).$$

As $x_{2i-3}, x_{2i-2} \in B_q(x_0, x_1) \cap \{XTx_n\}$, $\alpha^*(Sx_{2i-3}, x_{2i-3}) \geq 1$ and $\alpha^*(x_{2i-2}, Sx_{2i-2}) \geq 1$, then by (1), we have

$$q(x_{2i-1}, x_{2i-2}) \leq \mu(\max\{q(x_{2i-2}, x_{2i-3}), q(x_{2i-3}, x_{2i-2})\}).$$

As μ is non decreasing function, so

$$\mu^2(q(x_{2i-1}, x_{2i-2})) \leq \mu^2(\mu(\max\{q(x_{2i-2}, x_{2i-3}), q(x_{2i-3}, x_{2i-2})\})). \quad (9)$$

Combining inequalities (6), (8) and (9), we have

$$q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu^3 q(x_{2i-3}, x_{2i-2}), \mu^3 q(x_{2i-2}, x_{2i-3})\right\}. \quad (10)$$

Following the patterns of inequalities (4), (6) and (10), we have

$$q(x_{2i}, x_{2i+1}) \leq \max\left\{\mu^{2i}\left(q(x_0, x_1), \mu^{2i}(q(x_1, x_0))\right)\right\}. \quad (11)$$

Also, by Lemma 1.2, we have

$$q(x_{2i+1}, x_{2i}) \leq H_q(Tx_{2i}, Tx_{2i-1}).$$

As $x_{2i-1}, x_{2i} \in B_q(x_0, r) \cap \{XT(x_n)\}$, $\alpha^*(Sx_{2i-1}, x_{2i-1}) \geq 1$, and $\alpha^*(x_{2i}, Sx_{2i}) \geq 1$, then by (1), we have

$$q(x_{2i+1}, x_{2i}) \leq \mu(q(x_{2i-1}, x_{2i})), \quad (12)$$

which implies

$$q(x_{2i+1}, x_{2i}) \leq \max\{\mu(q(x_{2i-1}, x_{2i})), \mu(q(x_{2i}, x_{2i-1}))\}. \quad (13)$$

Using (5) in (12)

$$q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu^2(q(x_{2i-1}, x_{2i-2})), \mu^2(q(x_{2i-2}, x_{2i-1}))\right\}. \quad (14)$$

Combining the inequalities (8), (9) and (14), we have

$$q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu^3 q(x_{2i-3}, x_{2i-2}), \mu^3 q(x_{2i-2}, x_{2i-3})\right\}. \quad (15)$$

Following the patterns of inequalities (13), (14) and (15), we have

$$q(x_{2i+1}, x_{2i}) \leq \max\left\{\mu^{2i}(q(x_1, x_0)), \mu^{2i}(q(x_0, x_1))\right\}. \quad (16)$$

Now, by using the inequalities (11), (2) and triangle inequality, we have

$$q(x_0, x_{2i+1}) \leq \sum_{j=0}^{2i} \max\left\{\mu^j q(x_1, x_0), \mu^j q(x_0, x_1)\right\} < r. \tag{17}$$

Similarly, by using inequalities (16), (2) and triangle inequality, we have

$$q(x_{2i+1}, x_0) \leq \sum_{j=0}^{2i} \max\left\{\mu^j (q(x_1, x_0)), \mu^j (q(x_0, x_1))\right\} < r. \tag{18}$$

By inequality (17) and (18), we have $x_{2i+1} \in B_q(x_0, r)$. Also $q(x_{2i+1}, Tx_{2i+1}) = q(x_{2i+1}, x_{2i+2})$ and $q(Tx_{2i+1}, x_{2i+1}) = q(x_{2i+2}, x_{2i+1})$. As $\alpha^*(Sx_{2i+1}, x_{2i+1}) \geq 1$, so from condition (iii), we have $\alpha^*(x_{2i+2}, Sx_{2i+2}) \geq 1$. Similarly, we have

$$q(x_{2i+1}, x_{2i+2}) \leq \max\left\{\mu^{2i+1}(q(x_1, x_0)), \mu^{2i+1}(q(x_0, x_1))\right\}. \tag{19}$$

and

$$q(x_{2i+2}, x_{2i+1}) \leq \max\left\{\mu^{2i+1}(q(x_1, x_0)), \mu^{2i+1}(q(x_0, x_1))\right\}. \tag{20}$$

Also,

$$q(x_0, x_{2i+2}) \leq r \text{ and } q(x_{2i+2}, x_0) \leq r.$$

It following that $x_{2i+2} \in B_q(x_0, r)$. Also

$$q\left(x_{2i+2}, Tx_{2i+2}\right) = q\left(x_{2i+2}, x_{2i+3}\right) \text{ and } q\left(Tx_{2i+2}, x_{2i+2}\right) = q\left(x_{2i+3}, x_{2i+2}\right).$$

As $\alpha^*(x_{2i+2}, Sx_{2i+2}) \geq 1$, so from condition (iii) we have $\alpha^*(Sx_{2i+3}, x_{2i+3}) \geq 1$. Hence by mathematical induction $x_n \in B_q(x_0, r)$, $\alpha^*(x_{2n}, Sx_{2n}) \geq 1$ and $\alpha^*(Sx_{2n+1}, x_{2n+1}) \geq 1$, for all $n \in \mathbb{N}$. Also, $x_{2n} \in G(S)$. Now inequalities (11), (16), (19) and (20) can be written as

$$q(x_n, x_{n+1}) \leq \max\left\{\mu^n (q(x_1, x_0)), \mu^n (q(x_0, x_1))\right\}, \tag{21}$$

$$q(x_{n+1}, x_n) \leq \max\left\{\mu^n (q(x_1, x_0)), \mu^n (q(x_0, x_1))\right\}, \tag{22}$$

for all $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and let $k_1(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k \geq k_1(\varepsilon)} \max\left\{\mu^k (q(x_1, x_0)), \mu^k (q(x_0, x_1))\right\} < \varepsilon.$$

Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$, then

$$\begin{aligned} q(x_n, x_m) &\leq \sum_{k=n}^{m-1} q(x_k, x_{k+1}), \\ &\leq \sum_{k=n}^{m-1} \max\left\{\mu^k (q(x_1, x_0)), \mu^k (q(x_0, x_1))\right\}, \end{aligned}$$

$$q(x_n, x_m) < \sum_{k \geq k_1(\varepsilon)} \max\left\{\mu^n q(x_1, x_0), \mu^n q(x_0, x_1)\right\} < \varepsilon.$$

Thus we proved that $\{XT(x_n)\}$ is a left K- Cauchy sequence in (X, q). As (X, q) is left K sequentially complete, so $\{XT(x_n)\} \to x^* \in X$ and

$$\lim_{n\to\infty} q(x_{2n}, x^*) = \lim_{n\to\infty} q(x^*, x_{2n}) = 0. \tag{23}$$

As $\{x_{2n}\}$ is a subsequence of $\{XT(x_n)\}$, so $x_{2n} \to x^*$. Also, $\{x_{2n}\}$ is a sequence in $G(S)$ and $G(S)$ is closed, so $x^* \in G(S)$ and therefore

$$\alpha^*(x^*, Sx^*) \geq 1. \tag{24}$$

Now

$$q(x^*, x^*) \leq q(x^*, x_{2n}) + q(x_{2n}, x^*).$$

Which implies that $q(x^*, x^*) = 0$. Now, by Lemma 1.2, we have

$$q(x^*, Tx^*) \leq q(x^*, x_{2n+2}) + H_q(Tx_{2n+1}, Tx^*).$$

By assumption, inequality (1) holds for x^*. Also $\alpha^*(Sx_{2n+1}, x_{2n+1}) \geq 1$ and $\alpha^*(x^*, Sx^*) \geq 1$, so

$$q(x^*, Tx^*) \leq q(x^*, x_{2n+2}) + \mu\left(\max\left\{q(x_{2n+1}, x^*), q(x_{2n+1}, x_{2n+2}), \frac{q(x_{2n+1}, x_{2n+2})\, q(x_{2n+1}, Tx^*) + q(x^*, Tx^*)\, q_b(x^*, Tx_{2n+1})}{q(x_{2n+1}, Tx^*) + q(x^*, Tx_{2n+1})}\right\}\right).$$

Since $q_b(x^*, Tx_{2n+1}) \leq q_b(x^*, x_{2n+2})$. Taking limit as $n \to \infty$, on both side, we get

$$\lim_{n\to\infty} q_b(x^*, Tx_{2n+1}) = 0 \tag{25}$$

Letting $n \to \infty$, and by using inequalities (23) and (25), we obtain

$$q(x^*, Tx^*) = 0. \tag{26}$$

Now,

$$q(Tx^*, x^*) \leq H_q(Tx^*, Tx_{2n+1}) + q(x_{2n+2}, Tx^*).$$

As inequality (1) hold for x^*, $\alpha^*(x^*, Sx^*) \geq 1$ and $\alpha^*(Sx_{2n+1}, x_{2n+1}) \geq 1$, then

$$q(Tx^*, x^*) \leq \left(\max\left\{q(x_{2n+1}, x^*), q(x_{2n+1}, x_{2n+2}), \frac{q(x_{2n+1}, x_{2n+2})\, q(x_{2n+1}, Tx^*) + q(x^*, Tx^*)\, q_b(x^*, Tx_{2n+1})}{q(x_{2n+1}, Tx^*) + q(x^*, Tx_{2n+1})}\right\}\right) + q(x_{2n+2}, x^*).$$

Letting $n \to \infty$, and by using inequalities (23) and (26), we obtain

$$q(Tx^*, x^*) = 0. \tag{27}$$

From inequalities (26) and (27), we have $x^* \in Tx^*$. As $\alpha(x^*, Sx^*) \geq 1$ and $q(x^*, Tx^*) = q(Tx^*, x^*) = q(0, 0)$, then from (iii)

$$\alpha^*(Sx^*, x^*) \geq 1. \tag{28}$$

From (24) and (28), we have $\alpha^*(x^*, Sx^*) \geq 1$, $\alpha^*(Sx^*, x^*) \geq 1$. Thus implies $\alpha(x^*, y) \geq 1$, $\alpha(y, x^*) \geq 1$, for all $y \in Sx^*$. Thus by definition 1.9, $x^* = y$. Hence x^* is a common fixed point for S and T. □

Example 2.1 Let $X = [0, \infty)$ and $q(x, y) = \begin{cases} x + 2y & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$, for $(x, y) \in X \times X$, then (X, q) be left (right) K-sequentially complete quasi metric space. Consider μ be a function on $[0, \infty)$ define by $\mu(t) = \frac{3t}{4}$. Let $\mathcal{R}$ be the binary relation on X defined by

$$\begin{aligned} \mathcal{R} &= \left\{(x, \frac{x}{4}) : x \in \left\{0, 1, \frac{1}{16}, \frac{1}{256}, \frac{1}{4096}, ...\right\}\right\} \\ &\cup \left\{(\frac{x}{4}, x) : x \in \left\{\frac{1}{4}, \frac{1}{64}, \frac{1}{1024}, ...\right\}\right\}. \end{aligned}$$

Define the pair of multivalued mappings $T, S : X \to P(X)$ be

$$Tx = \begin{cases} \left[\frac{x}{4}, \frac{x}{2}\right] & \text{if } x \in [0, 1] \\ [x+1, x+2] & \text{if } x \in (1, \infty) \end{cases}, \quad Sx = \begin{cases} \{\frac{x}{4}\} & \text{if } x \in [0, 1] \\ \{2x\} & \text{if } x \in (1, \infty) \end{cases}.$$

Define $\alpha : X \times X \to [0, \infty)$ as follows

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \mathcal{R} \\ \frac{1}{2} & \text{if } x, y \in [0, 10) \wedge (x, y) \notin \mathcal{R} \\ 3 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} A &= \{x : \alpha^*(x, Sx) \geq 1\} = \left\{0, 1, \frac{1}{16}, \frac{1}{256}, \frac{1}{4096}, ...\right\}. \\ B &= \{y : \alpha^*(Sy, y) \geq 1\} = \left\{0, \frac{1}{4}, \frac{1}{64}, \frac{1}{1024}, ...\right\}. \end{aligned}$$

Let $x_0 = 1$ and $r = 21$, $B(x_0, r) = [0, 10)$. Then,

$$\begin{aligned} G(S) &= \{x : \alpha^*(x, Sx) \geq 1 \text{ and } x \in B_q(x_0, r)\} \\ &= \left\{0, 1, \frac{1}{16}, \frac{1}{256}, ...\right\}. \end{aligned}$$

Clearly $G(S)$ is closed and contained x_0, so condition (iv) is satisfied. Now, as $\frac{1}{4^{n-1}} \in B_q(x_0, r)$, for all $n \in \mathbb{N}$

$$q(\frac{1}{4^{n-1}}, T\frac{1}{4^{n-1}}) = q(\frac{1}{4^{n-1}}, \frac{1}{4 \times 4^{n-1}}).$$

and

$$q(T\frac{1}{4^{n-1}}, \frac{1}{4^{n-1}}) = q(\frac{1}{4 \times 4^{n-1}}, \frac{1}{4^{n-1}}).$$

As $\alpha^*\left(\frac{1}{4^{n-1}}, S\frac{1}{4^{n-1}}\right) \geq 1$, if n is odd, implies $\alpha^*\left(S\frac{1}{4 \times 4^{n-1}}, \frac{1}{4 \times 4^{n-1}}\right) \geq 1$, if n is odd. Also, $\alpha^*\left(S\frac{1}{4^{n-1}}, \frac{1}{4^{n-1}}\right) \geq 1$, if n is even, implies $\alpha^*\left(\frac{1}{4 \times 4^{n-1}}, S\frac{1}{4 \times 4^{n-1}}\right) \geq 1$, if n is even. Also, $0 \in B_{q_b}(x_0, r)$, $q_b(0, T0) = q_b(0, 0)$, $q_b(T0, 0) = q_b(0, 0)$. As $\alpha^*(0, S0) \geq 1$ if and only if $\alpha^*(S0, 0) \geq 1$. Hence, condition (iii) is satisfied. Now, $2, 3 \in B_q(x_0, r)$ with $\alpha^*(S3, 3) \ngeq 1, \alpha^*(2, S2) \ngeq 1$,

$$\max\{H_q(T2, T3), H_q(T3, T2)\} = \max\{11, 13\} = 13 \geq P_q(2, 3).$$

So, the contractive condition does not hold on whole $B_q(x_0, r)$. Now, for $11, 12 \in X$ with $\alpha^*(S11, 11) \geq 1, \alpha^*(12, S12) \geq 1$

$$\max\{H_q(T11, T12), H_q(T12, T11)\} = \max\{40, 38\} = 40 \geq P_q(x, y).$$

So, the contractive condition does not hold on X and $B_q(x_0, r)$. Now if $x, y \in B_q(x_0, r) \cap \{XTx_n\}$ with $\alpha^*(Sx, x) \geq 1$, $\alpha^*(y, Sy) \geq 1$, then in general $x = \frac{1}{4^{n-1}}$, $y = \frac{1}{4^{m-1}}$, where n is even, m is odd.

Case i: For $n \leq m$, we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[\frac{1}{4 \times 4^{n-1}}, \frac{1}{2 \times 4^{n-1}}\right], \left[\frac{1}{4 \times 4^{m-1}}, \frac{1}{2 \times 4^{m-1}}\right]\right) \\ &= \max\left\{q\left(\frac{1}{2 \times 4^{n-1}}, \frac{1}{4 \times 4^{m-1}}\right), q\left(\frac{1}{4 \times 4^{n-1}}, \frac{1}{2 \times 4^{m-1}})\right)\right\} \\ &= \max\left\{\frac{1}{2 \times 4^{n-1}} + \frac{1}{2 \times 4^{m-1}}, \frac{1}{4 \times 4^{n-1}} + \frac{1}{4^{m-1}})\right\} \\ &= \max\left\{\frac{4^{m-n} + 1}{2 \times 4^{m-1}}, \frac{4^{m-n} + 4}{4 \times 4^{m-1}}\right\} = \frac{4^{m-n} + 1}{2 \times 4^{m-1}}. \end{aligned}$$

Now,

$$\begin{aligned} H(Ty, Tx) &= \max\left\{\frac{1 + 4^{m-n}}{2 \times 4^{m-1}}, \frac{1 + 4 \times 4^{m-n}}{4 \times 4^{m-1}}\right\} \\ &= \frac{1 + 16 \times 4^{m-n}}{16 \times 4^{m-1}}. \end{aligned}$$

Now, we have

$$\frac{1+4\times 4^{m-n}}{4\times 4^{m-1}} < \frac{3}{4}\left(\frac{\left(\frac{3}{2\times 4^{n-1}}\right)\left(\frac{4\times 4^{m-n}+2}{4\times 4^{m-1}}\right)+\left(\frac{3}{2\times 4^{m-1}}\right)\left(\frac{4+2\times 4^{m-n}}{4\times 4^{m-1}}\right)}{\left(\frac{4\times 4^{m-n}+2}{4\times 4^{m-1}}\right)+\left(\frac{4+2\times 4^{m-n}}{4\times 4^{m-1}}\right)}\right),$$

$$\text{or } \max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\}\leq \mu\left(P_q(x,y)\right).$$

Case ii: Similarly, for $n > m$, we have

$$\begin{aligned}\max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\} &= \frac{1+4\times 4^{n-m}}{4\times 4^{n-1}}\\ &< \frac{3}{4}\left(\frac{1+2\times 4^{n-m}}{4^{n-1}}\right)=\mu\left(P_q(x,y)\right).\end{aligned}$$

Case iii: If $x=0$, $y=\frac{1}{4^{m-1}}$, we get

$$\begin{aligned}\max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\} &= \max\left\{\frac{1}{4^{m-1}},\frac{1}{2\times 4^{m-1}}\right\}=\frac{1}{4^{m-1}}\\ &< \frac{2}{4^{m-1}}=\mu\left(P_q(x,y)\right).\end{aligned}$$

Case iv: If $x=\frac{1}{4^{n-1}}$, $y=0$, we get

$$\max\left\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\right\}=\frac{1}{4^{n-1}}\leq \mu\left(P_q(x,y)\right).$$

Case v: Inequality (1) trivially holds for $x=0$ and $y=0$. Also,

$$\sum_{i=0}^{j}\max\left\{\mu^i\left(q\left(x_1,x_0\right)\right),\mu^i\left(q\left(x_0,x_1\right)\right)\right\}=9<21=r.$$

Hence, all the conditions of Theorem 2.1 are satisfied. Moreover, T and S have a common fixed point 0.

By taking complete metric space instead of left K-sequentially complete quasi metric space, we obtain the following result.

Theorem 2.2 *Let (X,d) be a complete metric space, $r>0$, $x_0\in X$, $S,T:X\to P\left(X\right)$ be the multivalued mappings on $B_d(x_0,r)$, $\mu\in\Psi$ and $\alpha:X\times X\to[0,+\infty)$. Suppose that the following assumptions hold:*

(i) for all $x,y\in B_d(x_0,r)\cap\{XT\left(x_n\right)\}$ with $\alpha^*\left(Sx,x\right)\geq 1,\alpha^*\left(y,Sy\right)\geq 1,$ we have

$$H_d\left(Tx,Ty\right)\leq \mu\left(P_d\left(x,y\right)\right),$$

(ii) $\sum_{i=0}^{j}\mu^i\left(d\left(x_1,x_0\right)\right)<r$, for all $j\in N\cup\{0\}$.

(iii) if $x\in B_d(x_0,r)$, $d\left(x,Tx\right)=d\left(x,y\right)$, then

(a) $\alpha^*\left(x,Sx\right)\geq 1$, implies $\alpha^*\left(Sy,y\right)\geq 1$,

(b) $\alpha^*\left(Sx,x\right)\geq 1$, implies $\alpha^*\left(y,Sy\right)\geq 1$,

(iv) the set $G(S)=\{x:\alpha\left(x,Sx\right)\geq 1 \text{ and } x\in B_d(x_0,r)\}$ is closed and contained x_0. Then the subsequence $\{x_{2n}\}$ of $\{XT\left(x_n\right)\}$ is a sequence in $G\left(S\right)$ and a sequence $\{x_{2n}\}\to x^*\in G(S)$ and $d\left(x^*,x^*\right)=0$. Also, if inequality (i) holds for x^*. Then T and S have a common fixed point x^* in $B_d(x_0,r)$.

By excluding open ball, we obtain the following result.

Theorem 2.3 *Let (X,q) be a complete left K-sequentially quasi metric space, $\alpha:X\times X\to[0,+\infty)$, $\mu\in\Psi$, $x_0\in X$ and $S,T:X\to P\left(X\right)$. Suppose that the following assertions hold:*

(i) For all $x,y\in X\cap\{XT\left(x_n\right)\}$ with $\alpha^*\left(Sx,x\right)\geq 1,\alpha^*\left(y,Sy\right)\geq 1,$ we have

$$\max\{H_q\left(Tx,Ty\right),H_q\left(Ty,Tx\right)\}\leq \mu\left(P_q\left(x,y\right)\right),$$

where

$$P_q(x,y) = \max\left\{q(x,y), q(x,Tx), \frac{q(x,Tx)\,q(x,Ty) + q(y,Ty)\,q(y,Tx)}{q(x,Ty)+q(y,Tx)}\right\}.$$

(ii) If $q(x,Tx) = q(x,y)$ *and* $q(Tx,x) = q(y,x)$, *then*

(a) $\alpha^*(x,Sx) \geq 1$, *implies* $\alpha^*(Sy,y) \geq 1$, *(b)* $\alpha^*(Sx,x) \geq 1$, *implies* $\alpha^*(y,Sy) \geq 1$,

(iii) The set $G(S) = \{x : \alpha(x,Sx) \geq 1\}$ *is closed and contained* x_0.

Then the subsequence $\{x_{2n}\}$ *of* $\{XT(x_n)\}$ *is a sequence in* $G(S)$ *and a sequence* $\{x_{2n}\} \to x^* \in G(S)$ *and* $q(x^*,x^*) = 0$. *Also, if inequality (i) holds for* x^*. *Then* T *and* S *have a common fixed point* x^* *in* X.

By taking self mappings, we obtain the following result.

Theorem 2.4 *Let* (X,q) *be a complete left* K*-sequentially quasi metric space,* $\alpha : X \times X \to [0,+\infty)$ *be a function,* $r > 0$, $\mu \in \Psi$, $S,T : X \to X$ *be the self mappings,* $x_0 \in X$ *and* $x_n = Tx_{n-1}$ *be a Picard sequence. Suppose that the following assumptions hold:*

(i) For all $x,y \in B_q(x_0,r) \cap \{x_n\}$ *with* $\alpha(Sx,x) \geq 1$ *and* $\alpha(y,Sy) \geq 1$, *we have*

$$\max\{q(Tx,Ty), q(Ty,Tx)\} \leq \mu(P_q(x,y)),$$

where

$$P_q(x,y) = \max\left\{q(x,y), q(x,Tx), \frac{q(x,Tx)\,q(x,Ty) + q(y,Ty)\,q(y,Tx)}{q(x,Ty)+q(y,Tx)}\right\}.$$

(ii) $\sum_{i=0}^{j} \max\left\{\mu^i(q(x_1,x_0)), \mu^i(q(x_0,x_1))\right\} < r$, *for all* $j \in N \cup \{0\}$.

(iii) if $x \in B_q(x_0,r)$, *then*

(a) $\alpha(x,y) \geq 1$ *implies* $\alpha(STx,x) \geq 1$,

(b) $\alpha^*(y,x) \geq 1$ *implies* $\alpha(Tx,STx) \geq 1$,

(vi) the set $G(S) = \{x : \alpha(x,y) \geq 1 \text{ and } x \in B_q(x_0,r)\}$ *is closed and contained* x_0.

Then the subsequence $\{x_{2n}\}$ *of* $\{x_n\}$ *is a sequence in* $G(S)$ *and a sequence* $\{x_{2n}\} \to x^* \in G(S)$ *and* $q(x^*,x^*) = 0$. *Also, if inequality (i) holds for* x^*. *Then* T *and* S *have a common fixed point* x^* *in* $B_q(x_0,r)$.

3. **Fixed point results for contractions endowed with a graph and a partial order**

Consistent with Jachymski [11], let (X,q) be a quasi metric space and $\triangle$ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e, $\triangle \subseteq E(G)$. We assume G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph [16], by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length $m (m \in N)$ is a sequence $\{x_i\}_{i=0}^m$ of $m+1$ vertices such that $x_0 = x$, $x_m = y$ and $(x_{n-1},x_n) \in E(G)$ for $i = 1,...,m$.

Definition 3.1 Let (X,q) be a quasi metric space endowed with a graph G and $S,T : X \to P(X)$ be multivalued mappings. Assume that for $r > 0$, $x_0 \in B_q(x_0,r)$ and $\mu \in \Psi$, the following conditions hold:

$$\max\{H_q(Tx,Ty), H_q(Ty,Tx)\} \leq \mu(P_q(x,y)), \tag{29}$$

for all $x,y \in B_q(x_0,r) \cap \{XT(x_n)\}$ with $\{(y,v) \in E(G) : v \in Sy\}$ and $\{(u,x) \in E(G), u \in Sx\}$, where

$$P_q(x,y) = \max\left\{q(x,y), q(x,Tx), \frac{q(x,Tx)\,q(x,Ty) + q(y,Ty)\,q(y,Tx)}{q(x,Ty)+q(y,Tx)}\right\}.$$

Then the mappings (S,T) are called a $\mu-$graphic contractive multivalued mappings on open ball.

Theorem 3.1 *Let (X,q) be a complete left K sequentially quasi metric space endowed with graph G, Let $r>0$, $x_0 \in B_q(x_0,r)$ and S,$T : X \to P(X)$ be $\mu-$graphic contractive multivalued mappings on $B_q(x_0,r)$. Suppose that the following assumptions hold:*

(i) $\sum\limits_{i=0}^{j} \max\left\{\mu^i q(x_1,x_0), \mu^i q(x_0,x_1)\right\} < r$, *for all* $j \in N \cup \{0\}$.

(ii) If $x \in B_q(x_0,r)$, $q(x,Tx) = q(x,y)$ *and* $q(Tx,x) = q(y,x)$, *then*

(a) $(x,u) \in E(G)$, *for all* $u \in Sx$ *implies* $(v,y) \in E(G)$, *for all* $v \in Sy$,

(b) $(u,x) \in E(G)$, *for all* $u \in Sx$ *implies* $(y,v) \in E(G)$, *for all* $v \in Sy$.

(iii) The set $G(S) = \{x : (x,y) \in E(G) \text{ for all } y \in Sx \text{ and } x \in B_q(x_0,r)\}$ *is closed and contained* x_0.

Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$ and a sequence $\{x_{2n}\} \to x^* \in G(S)$. Also, if inequality (29) holds for x^*. Then T and S have a common fixed point x^* in $B_q(x_0,r)$.

Proof. Define $\alpha : X \times X \to [0,\infty)$, by $\alpha(y,v) = 1$, for all $v \in Sy$, if and only if $y \in B_q(x_0,r) \cap \{XT(x_n)\}$ with $\{(y,v) \in E(G) : v \in Sy\}$. Also $\alpha(u,x) = 1$, for all $u \in Sx$, if and only if $x \in B_q(x_0,r) \cap \{XT(x_n)\}$ with $\{(u,x) \in E(G), u \in Sx\}$. Moreover $\alpha(x,y) = 0$, otherwise. Now, as (S,T) is a $\mu-$graphic contractive multivalued mappings on open ball, so inequality (29), implies inequality (1). Assumption (i) of Theorem 3.1 implies assumption (ii) of Theorem 2.1. Assumption (ii) of Theorem 3.2 implies assumption (iii) of Theorem 2.1. Assumption (iii) of Theorem 3.2 implies assumption (iv) of Theorem 2.1. So, all conditions of Theorem 2.1 are satisfied. Hence the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$, for all $n \in N \cup \{0\}$ and a sequence $\{x_{2n}\} \to x^* \in G(s)$. Also if inequality (29) holds for x^*, then inequality (1) holds for x^*. Then T and S have a common fixed point x^* in $B_q(x_0,r)$. □

Definition 3.2 [16] Let $(X,\preceq)$ be a partially ordered set and $T : X \to X$ be a given mapping. We say that T is non decreasing with respect to $\preceq$ if $x,y \in X$, $x \preceq y \Rightarrow Tx \preceq Ty$.

Definition 3.3 [16] Let $(X,\preceq)$ be a partially ordered set and d be a metric on X. We say that $(X,\preceq,d)$ is regular if for every nondecreasing sequence $\{x_n\} \subset X$ such that $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

Now, we have the following results.

Theorem 3.2 *Let $(X,\preceq,q)$ be an ordered complete left K sequentially quasi metric space, $r>0$, $x_0 \in X$ and S,$T : X \to P(X)$ be a non decreasing mappings on $B_q(x_0,r)$, with respect to $\preceq$ and suppose there exists a function $\mu \in \Psi$ satisfy the following:*

(i) for all $(x,y) \in B_q(x_0,r) \cap \{XT(x_n)\}$ *with* $Sx \preceq x$ *and* $y \preceq Sy$, *we have*

$$\max\{H_q(Tx,Ty), H_q(Ty,Tx)\} \le \mu(P_q(x,y)), \tag{30}$$

where

$$P_q(x,y) = \max\left\{q(x,y), q(x,Tx), \frac{q(x,Tx)\,q(x,Ty) + q(y,Ty)\,q(y,Tx)}{q(x,Ty)+q(y,Tx)}\right\}.$$

(ii) $\sum\limits_{i=0}^{j} \max\left\{\mu^i q(x_1,x_0), \mu^i q(x_0,x_1)\right\} < r$, *for all* $j \in N \cup \{0\}$.

(iii) If $x \in B_q(x_0,r)$, $q(x,Tx) = q(x,y)$ *and* $q(Tx,x) = q(y,x)$, *then*

(a) $x \preceq Sx$, *implies* $Sy \preceq y$, *(b)* $Sx \preceq x$, *implies* $y \preceq Sy$.

(iv) The set $G(S) = \{x : x \preceq Sx \text{ and } x \in B_q(x_0,r)\}$ *is closed and contained* x_0.

Then the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$, for all $n \in N \cup \{0\}$ and $\{x_{2n}\} \to x^* \in G(S)$. Also if inequality (i) holds for x^*. Then T and S have a common fixed point x^* in $B_q(x_0,r)$.

Proof. Define $\alpha : X \times X \to [0, \infty)$, by $\alpha(y, v) = 1$, for all $v \in Sy$, if and only if $y \in B_q(x_0, r) \cap \{XT(x_n)\}$ with $y \preceq v$, $v \in Sx$. Also $\alpha(u, x) = 1$, for all $u \in Sx$, if and only if $x \in B_q(x_0, r) \cap \{XT(x_n)\}$ with $x \succeq u$, $u \in Sx$. Moreover $\alpha(x, y) = 0$, otherwise. Then, clearly Assumption (i)-(iv) of Theorem 3.2 implies assumption (i)-(iv) of Theorem 2.1. Hence the subsequence $\{x_{2n}\}$ of $\{XT(x_n)\}$ is a sequence in $G(S)$, for all $n \in N \cup \{0\}$ and a sequence $\{x_{2n}\} \to x^* \in G(s)$. Also, if inequality (i) holds for x^*, then inequality (2.1) holds for x^*. Then T and S have a common fixed point x^* in $B_q(x_0, r)$. □

Remark 3.1. By taking six proper subsets of $P_q(x, y)$ instead of $P_q(x, y)$, we can obtain six new corollaries for each of theorems; Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 3.1 and Theorem 3.2.

Remark 3.2. Fixed point result in right K sequentially quasi metric space can be obtained in a similar way.

4. **Conclusion**

In the present paper, we have obtained sufficient conditions to ensure the existence of fixed point for a pair of multivalued mappings satisfying $\mu - \alpha^*$ contractive condition on a sequence contained in an open ball. An example is given to demonstrate the variety of our results. Fixed point results with graphic contractions on a sequence contained in an open ball for such mappings are also established. Results endowed with a partial order have been obtained. Moreover, we investigate our results in a better framework of quasi-metric spaces.

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